MAT159 Test Solutions – Test #10

Mustafa Motiwala

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Question 1

Prove that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

Solution.

Pick some $\frac{1}{2} < r < 1$. Taking $f_n(x) = x^n$ on the domain $[-r, r]$, we have the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f(x) = 1/(1-x)$ by the Weierstrass M-test (since each $\|f_n\| \le r^n$ and $\sum_n r^n$ converges as $|r| < 1$).

Moreover, we have $f_n'(x) = nx^{n-1}$ and again by the Weierstrass M-test we have the f_n' uniformly convergent (here, $\|f_n'\| \le nr^{n-1}$ and the series $\sum_n nr^{n-1}$ can be seen to converge by the ratio test).

So, we can differentiate term by term and we obtain

$$\sum_{n=1}^{\infty} f_n' = f'(x) \Longrightarrow \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

Multiplying by x, we have

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

The above identity is valid for all $x \in [-r, r]$, so substituting $x = 1/2$, we get

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$$

as needed. ■

Question 2

Prove that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \ln 2$$

Solution.

First, observe that

$$\frac{1}{(2n+1)(2n+2)} = \frac{1}{2n+1} - \frac{1}{2n+2}$$

so

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \sum_{n=0}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

We will evaluate this series. Naively, we might try

$$\begin{aligned}
\ln(2) &= \int_0^1 \frac{1}{1+x}\,\mathrm{d}x \\
&= \int_0^1 \left(\sum_{n=0}^{\infty} (-x)^n\right) \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \left(\int_0^1 (-x)^n \,\mathrm{d}x\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}
\end{aligned}$$

but exchanging the integral and the series cannot be justified as the series convergence is not uniform. However, we will see that the integral is still preserved in the limit. That is, we

Claim. $\displaystyle\int_0^1 \frac{1}{1+x}\,\mathrm{d}x = \lim_{N\to\infty} \sum_{n=0}^{N} \left(\int_0^1 (-x)^n \,\mathrm{d}x\right).$

Proof. Let $S_N(x) = \sum_{n=0}^{N} (-x)^n$; then, for $x \in [0,1]$, we have

$$\begin{aligned}
\left|\frac{1}{1+x} - S_N(x)\right| &= \left|\frac{1}{1+x} - \frac{1-(-x)^{N+1}}{1+x}\right| \\
&= \frac{x^{N+1}}{1+x} \\
&\leq x^{N+1}
\end{aligned}$$

so that

$$\begin{aligned}
\left|\int_0^1 \frac{1}{1+x}\,\mathrm{d}x - \int_0^1 S_N(x)\,\mathrm{d}x\right| &\leq \int_0^1 x^{N+1}\,\mathrm{d}x \\
&= \frac{1}{N+2}
\end{aligned}$$

so $\displaystyle\lim_{N\to\infty} \int_0^1 S_N(x)\,\mathrm{d}x = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x$ but by linearity of the integral over finitely many terms, this is exactly what we want. ■

Combined with the not so naive now argument above, this gives us the claimed equality. ■

Question 3

Let K be a compact set. Let $f : K \to \mathbb{R}$ be continuous and $f_n : K \to \mathbb{R}$ be a sequence of continuous functions such that $\forall x \in K, f_n(x) \searrow f(x)$. Prove that $f_n \rightrightarrows f$.

Solution.

Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, define $g_n = f_n - f \geq 0$ and let

$$E_n = \{x \in K : g_n(x) < \varepsilon\} = g^{-1}((-\infty, \varepsilon))$$

Since f_n, f are continuous, so is g and hence each E_n is open. Moreover, for every $x \in K$, $f_n(x) \to f$ so $x \in E_m$ for some sufficiently large m (note that this m depends on x). Thus, the family $\{E_n : n \in \mathbb{N}\}$ forms an open cover of K. By compactness, there is a finite subcover $E_{n_1}, ..., E_{n_k}$.

Now, since $f_{n+1}(x) \leq f_n(x)$, the sets E_n are increasing. So, the finite subcover reduces to a single E_N where $N = \max(n_1, ..., n_k)$. That is, $E_N = K$. For any $n > N$, we have $K \subseteq E_N \subseteq E_n$ so $E_n = K$ and thus for all $x \in K = E_n$ we have $|f_n(x) - f(x)| = f_n(x) - f(x) < \varepsilon$ as needed. ■

Remark. This is known as Dini's theorem. Of course, we could also assume $f_n(x) \nearrow f(x)$ instead.