The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of a test and/or for revealing the test materials to other students or to any unauthorized institution.

Please note, once this test has begun, you ***CANNOT*** *re-write it.* Let $\varphi\colon [a,b] \to \mathbb{R}$ be a convex function and let $f\colon [a,b] \to [a,b]$ be Riemann integrable.

Definition 0.1 (Convex Function)**.** A function $\varphi : [a, b] \to \mathbb{R}$ is convex if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

Question 1 (5 points)

Let $\varphi : [a, b] \to \mathbb{R}$ be convex. Let $c \in (a, b)$. Show that there exists a constant $\mu \in \mathbb{R}$ such that

$$\varphi(x) \geq \varphi(c) + \mu(x - c) \qquad \text{for all } x \in [a, b].$$

Question 2 (5 points)

Prove that for a function $\phi \in \mathcal{C}^2[a, b]$, ϕ is convex in $[a, b]$ if and only if $\forall x \in (a, b)$, $\phi^{(2)}(x) \geq 0$.

Question 3 (5 points)

Let $x_1, \ldots, x_n \in [a, b]$ and let $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. Prove that

$$\varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i \, \varphi(x_i).$$

Question 4 (5 points)

Let $f \in \mathfrak{R}[a,b]$, and assume that the function $\phi : \mathbb{R} \to \mathbb{R}$ is convex. Show that $\phi \circ f \in \mathfrak{R}[a,b]$, and prove Jensen's inequality:

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right) \leq \frac{1}{b-a}\int_a^b \varphi(f(x))\,\mathrm{d}x.$$

Question 5 (5 points)
Assume that $f \in \mathcal{C}^1[a,b]$ and $f(a) = 0$. Prove that

$$(\sup_{x\in[a,b]} |f(x)|)^2 \leq (b-a)\int_a^b (f'(x))^2 dx.$$

Hint Notice that $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(t) = t^2$ is a convex function.

Question 6 (5 points)
Suppose $f, g \in \mathcal{C}[0,1]$, and let $p, q > 1$ with $1/p + 1/q = 1$. Prove the Hölder inequality:

$$\int_0^1 |f(x)g(x)| \,\mathrm{d}x \leq \left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{1/p} \left(\int_0^1 |g(x)|^q \,\mathrm{d}x\right)^{1/q}$$

Hint Apply Jensen's inequality to $\varphi(x) = -\log x$, and prove the following lemma first:

Lemma : Let $a, b \geq 0, p, q > 1$ with $1/p + 1/q = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

From there, choose a, b appropriately.