MAT159 Test Solutions – Mock Exam 1

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2026-04-02

Definition. φ is convex if for all $x, y \in \operatorname{dom}(\varphi)$ and $\lambda \in [0,1]$, we have

$$\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

Question 1

Let $\varphi : [a,b] \to \mathbb{R}$ be convex and $c \in (a,b)$. Show that there exists a constant $\mu \in \mathbb{R}$ such that

$$\varphi(x) \ge \varphi(c) + \mu(x-c) \qquad \forall x \in [a,b]$$

Solution. Let

$$L = \sup_{a \le x < c} \frac{\varphi(c)-\varphi(x)}{c-x} \qquad \text{and} \qquad R = \inf_{c < x \le b} \frac{\varphi(x)-\varphi(c)}{x-c}$$

We will show $L \le R$ (which will also show that both numbers exist). It suffices to show for any $u, v \in [a,b]$ with $u < c < v$ we have

$$\frac{\varphi(c)-\varphi(u)}{c-u} \le \frac{\varphi(v)-\varphi(c)}{v-c}$$

Taking $t = (c-u)/(v-u) \in [0,1]$, we obtain

$$c = u + (c-u) = u + t(v-u) = (1-t)u + tv$$

so by convexity $\varphi(c) \le (1-t)\varphi(u) + t\varphi(v)$ which implies

$$\varphi(c) - \varphi(u) \le t(\varphi(v)-\varphi(u)) = (c-u)\frac{\varphi(v)-\varphi(u)}{v-u}$$

and

$$\varphi(v) - \varphi(c) \ge (1-t)(\varphi(v)-\varphi(u)) = (v-c)\frac{\varphi(v)-\varphi(u)}{v-u}$$

so that

$$\frac{\varphi(c)-\varphi(u)}{c-u} \le \frac{\varphi(v)-\varphi(u)}{v-u} \le \frac{\varphi(v)-\varphi(c)}{v-c}$$

which gives the desired inequality.

Thus, L, R both exist and $L \le R$. We take μ to be any element of $[L, R]$. Let $x \in [a,b]$.

Of course, if $x = c$, we have $\varphi(x) \ge \varphi(c) + \mu \cdot 0$. If $x < c$, then

$$\frac{\varphi(c)-\varphi(x)}{c-x} \le L \le \mu \Longrightarrow \varphi(c) \le \varphi(x) + \mu(c-x) \Longrightarrow \varphi(x) \ge \varphi(c) + \mu(x-c)$$

If $x > c$, we have

$$\frac{\varphi(x)-\varphi(c)}{x-c} \ge R \ge \mu \Longrightarrow \varphi(x) \ge \varphi(c) + \mu(x-c)$$

In every case, we have $\varphi(x) \ge \varphi(c) + \mu(x-c)$, as needed. ∎

Question 2

Let $\varphi \in \mathcal{C}^2[a,b]$. Prove that φ is convex if and only if $\forall x \in (a,b), \varphi^{(2)}(x) \ge 0$.

Solution.

($\Longrightarrow$) Suppose φ is convex. Fix $x \in (a,b)$; we have, by Taylor's theorem, that for sufficiently small h

$$\varphi(x+h) = \varphi(x) + \varphi^{(1)}(x)h + \varphi^{(2)}(x)h^2 + o(h^2)$$

and

$$\varphi(x-h) = \varphi(x) - \varphi^{(1)}(x)h + \varphi^{(2)}(x)h^2 + o(h^2)$$

so

$$\varphi^{(2)}(x) = \lim_{h \to 0} \frac{\varphi(x+h) + \varphi(x-h) - 2\varphi(x)}{h^2}$$

We claim that the quotient in the limit is always ≥ 0 for sufficiently small h. Indeed, we have

$$x = \frac{1}{2}(x+h) + \frac{1}{2}(x-h)$$

so

$$\varphi(x) \leq \frac{1}{2}\varphi(x+h) + \frac{1}{2}\varphi(x-h)$$

and thus

$$\varphi(x+h) - \varphi(x-h) - 2\varphi(x) \geq 0$$

So, $\varphi^{(2)}(x)$ is the limit of non-negative values and is hence itself non-negative. ■

(⟸) Suppose $\varphi^{(2)}(x) \geq 0$ for all $x \in (a,b)$. Let $x, y \in [a,b]$ and $\lambda \in [0,1]$. We want to show

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

If $x = y$, the inequality is trivial. If $x < y$, let $c = \lambda x + (1-\lambda)y \in (x,y)$. By Taylor's theorem (in the Lagrange form of the remainder), there exists $\xi \in (x,c)$ and $\zeta \in (c,y)$ such that

$$\varphi(x) = \varphi(c) + \varphi^{(1)}(c)(x-c) + \frac{\varphi^{(2)}(\xi)}{2}(x-c)^2$$

and

$$\varphi(y) = \varphi(c) + \varphi^{(1)}(c)(y-c) + \frac{\varphi^{(2)}(\zeta)}{2}(y-c)^2$$

Applying the assumption $\varphi^{(2)} \geq 0$ and some algebra on $x-c, y-c$, we obtain

$$\varphi(x) \geq \varphi(c) + \varphi^{(1)}(c)((1-\lambda)(x-y))$$

and

$$\varphi(y) \geq \varphi(c) + \varphi^{(1)}(c)(\lambda(y-x))$$

from which we obtain (by multiplying the first inequality by λ, the second by $1-\lambda$, and adding)

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) \geq \varphi(c) = \varphi(\lambda x + (1-\lambda)y)$$

which is exactly the desired inequality. ■

Question 3

Let $x_1, ..., x_n \in [a,b]$ and $\lambda_1, ..., \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$. Prove that

$$\varphi\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i \varphi(x_i)$$

Solution. The case for $n = 1$ is trivial and the case for $n = 2$ is simply the definition of convexity.

Now, suppose the result holds for n. Let $x_1, ..., x_{n+1} \in [a,b]$ and $\lambda_1, ..., \lambda_{n+1} > 0$ with $\sum_{i=1}^{n+1} \lambda_i = 1$.

Observe that

$$\begin{aligned}\sum_{i=1}^{n+1}\lambda_i x_i &= \sum_{i=1}^{n}\lambda_i x_i + \lambda_{n+1}x_{n+1}\\ &= (1-\lambda_{n+1})\sum_{i=1}^{n}\mu_i x_i + \lambda_{n+1}x_{n+1} \qquad \text{where } \mu_i = \frac{\lambda_i}{1-\lambda_{n+1}}\end{aligned}$$

so, by definition of convexity, we have

$$\varphi\left(\sum_{i=1}^{n+1}\lambda_i x_i\right) \le (1-\lambda_{n+1})\varphi\left(\sum_{i=1}^{n}\mu_i x_i\right) + \lambda_{n+1}\varphi(x_{n+1})$$

Applying the induction hypothesis allows us to conclude. ■

Question 4

Let $f \in \mathfrak{R}[a,b]$ and assume $\varphi : \mathbb{R} \to \mathbb{R}$ is convex. Show that $\varphi \circ f \in \mathfrak{R}[a,b]$ and

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right) \le \frac{1}{b-a}\int_a^b \varphi(f(x))\,\mathrm{d}x$$

Solution. The function $\varphi \circ f$ is integrable because φ is continuous and f is integrable. For any partition $(\Gamma, \eta) \in \Omega^*[a,b]$ with $\Gamma = x_0 < \cdots < x_n$ we have

$$\begin{aligned}\varphi\left(\sigma\left(\frac{1}{b-a}f, \Gamma, \eta\right)\right) &= \varphi\left(\sum_{i=0}^{n-1} f(\eta_i)\frac{x_{i+1}-x_i}{b-a}\right)\\ &\le \sum_{i=1}^{n-1}\varphi(f(\eta_i))\frac{x_{i+1}-x_i}{b-a}\\ &= \sigma\left(\frac{1}{b-a}(\varphi\circ f), \Gamma, \eta\right)\end{aligned}$$

so taking limits (again, φ is continuous) we have

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right) \le \frac{1}{b-a}\int_a^b \varphi(f(x))\,\mathrm{d}x$$

as needed.

Question 5

Assume that $f \in \mathcal{C}^1[a,b]$ and $f(a) = 0$. Prove that

$$\left(\sup_{x\in[a,b]}|f(x)|\right)^2 \le (b-a)\int_a^b (f'(x))^2\,\mathrm{d}x$$

Solution. The function $\varphi(t) = t^2$ is convex, so we have by Q4 that for all $x \in [a,b]$,

$$\left(\frac{1}{x-a}\int_a^x f'(x)\,\mathrm{d}x\right)^2 \le \frac{1}{x-a}\int_a^x (f'(x))^2\,\mathrm{d}x$$

which, by the FTC is equivalent to

$$f(x)^2 \le (x-a)\int_a^x (f'(x))^2\,\mathrm{d}x$$

Since $(x-a) \le (b-a)$ and $(f'(x))^2 \ge 0$, we can relax the inequality to

$$f(x)^2 = |f(x)|^2 \le (b-a)\int_a^b (f'(x))^2\,\mathrm{d}x$$

Since this is true for all $x \in [a,b]$, we get

$$\sup_{x\in[a,b]} |f(x)|^2 \le (b-a)\int_a^b (f'(x))^2\,\mathrm{d}x$$

Since $|f| \ge 0$, we have $\sup_{x\in[a,b]}|f(x)|^2 = \left(\sup_{x\in[a,b]}|f(x)|\right)^2$ so the above is the desired inequality. ■

Question 6

Suppose $f, g \in \mathcal{C}[0,1]$ and let $p, q > 1$ with $1/p + 1/q = 1$. Prove the Hölder inequality:

$$\int_0^1 |f(x)g(x)|\,\mathrm{d}x \le \left(\int_0^1 |f(x)|^p\,\mathrm{d}x\right)^{1/p}\left(\int_0^1 |g(x)|^q\,\mathrm{d}x\right)^{1/q}$$

Solution. We first prove the following lemma (which is known as Young's inequality for products).

Lemma. *Let $a, b \ge 0$ and $p, q > 1$ with $1/p + 1/q = 1$. Then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.*

Proof. If $a = 0$ or $b = 0$, we are done. Assume $a, b > 0$, then we have

$$\begin{aligned} -\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) &\le -\frac{1}{p}\log(a^p) - \frac{1}{q}\log(b^q) \\ &= -\log(a) - \log(b) \\ &= -\log(ab) \end{aligned}$$

which gives $\log(ab) \le \log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)$ and exponentiating gives the result. ■

Now, set

$$A = \left(\int_0^1 |f(x)|^p\,\mathrm{d}x\right)^{1/p} \qquad \text{and} \qquad B = \left(\int_0^1 |g(x)|^q\,\mathrm{d}x\right)^{1/q}$$

so that, by Young's inequality, we have for all $x \in [0,1]$ that

$$\frac{|f(x)|}{A}\frac{|g(x)|}{B} \le \frac{|f(x)|^p}{pA^p} + \frac{|g(x)|^q}{qB^q}$$

Integrating both sides gives

$$\frac{1}{AB}\int_0^1 |f(x)g(x)|\,\mathrm{d}x \le \frac{1}{p} + \frac{1}{q} = 1$$

which is what we want. ■