The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of a test and/or for revealing the test materials to other students or to any unauthorized institution.

Please note, once this test has begun, you ***CANNOT*** *re-write it.*

Definition 0.1 (Asymptotic equivalence)**.** Let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$. We say $f(n)$ is *asymptotically equivalent* to $g(n)$ as $n \to \infty$, denoted by $f(n) \sim g(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Example

- $n^2 \sim n^2 + n$, since
$$\lim_{n\to\infty} \frac{n^2}{n^2 + n} = 1.$$
- $3^n + 2^n \sim 3^n + n^{159}$, since
$$\lim_{n\to\infty} \frac{3^n + 2^n}{3^n + n^{159}} = 1.$$

Question 1 (5 points)
Show that

$$\int_1^n \ln x dx \sim \int_2^{n+1} \ln x dx$$

Hint Show that

$$\lim_{n\to\infty} \frac{\int_2^{n+1} \ln x dx - \int_1^n \ln x dx}{\int_2^{n+1} \ln x dx} = 0.$$

Question 2 (5 points)

Show that

$$\ln(n!) \sim n \ln n - n.$$

Hint Write

$$\ln(n!) = \sum_{k=1}^{n} \ln k,$$

from there, show that

$$\int_1^n \ln x dx < \ln(n!) < \int_2^{n+1} \ln x dx.$$

To show the left part of this inequality, write

$$\int_1^n \ln x dx = \sum_{k=1}^{n-1} \int_k^{k+1} \ln x dx,$$

and use the fact that $\ln x$ is monotonically increasing. Similarly for the right part.

(cont)

Question 3 (5 points)

Find the Taylor series of $\ln(1+x)$, centered at 0. Indicate its radius of convergence, and show that the Taylor series converges to $\ln(1+x)$ for any $x \in (-1, 1)$.

Hint First find the Taylor series, then use the ratio test to find the radius of convergence. Finally, estimate the remainder. For $x > 0$, use the Lagrange remainder. For $x < 0$, use the Cauchy remainder. If you forget what does the Cauchy remainder look like, tell Mustafa "Thank you for your great support along the semester.", then he will help you recall what is the Cauchy remainder (hopefully).

(cont)

Question 4 (5 points)

Let

$$\ln(n!) \;=\; n\ln n - n + \tfrac{1}{2}\ln n + C_n.$$

Show that $\lim_{n\to\infty} C_n$ exists.

Hint To solve this, verify by direct computation that (be patient, you can do this, it is just elementary algebraic manipulation, calm down and make the work done!)

$$C_{n+1} - C_n = 1 - n\ln\frac{n+1}{n} - \frac{1}{2}\ln\frac{n+1}{n}.$$

now expand $\ln\frac{n+1}{n} = \ln 1 + \frac{1}{n}$ using the Taylor series of $\ln 1 + x$ obtained in the previous question. If everything goes well, you should now arrive at

$$C_{n+1} - C_n = -\frac{1}{12n^2} + o(n^{-2}).$$

From there, conclude that C_n is decreasing, moreover, $\lim_{n\to\infty} C_n$ exists.

(cont)

Question 5 (5 points)

Show that

$$\lim_{n\to\infty} C_n = \frac{1}{2}\ln 2\pi.$$

Hint Recall the Wallis Formula (use it for free)

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n)!^2(2n+1)}.$$

Now we already know that C_n converges to some real number, say, C. By the definition of C_n, define K_n as

$$K_n = e^{C_n} = \frac{n!e^n}{n^{n+\frac{1}{2}}}.$$

From there, $n! = e^{-n}n^{n+\frac{1}{2}}K_n$. Plug into the Wallis formula, and find $K = \lim_{n\to\infty} K_n$. Finally, find $C = \ln K$.

(cont)

Question 6 (5 points)

Prove the Stirling formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Hint No more hint, not any more... I have carried you till this far, now you need to finish this fantastic part by yourself...Enjoy, and good luck.

(cont)

Scratch page; work here will not be graded unless indicated otherwise.