MAT159 Test Solutions – Mock Exam 2

Mustafa Motiwala

2026-04-08

Question 1

Show that

$$\int_1^n \ln x \, \mathrm{d}x \sim \int_2^{n+1} \ln x \, \mathrm{d}x$$

Solution. It suffices to show

$$\lim_{n\to\infty} \frac{\int_2^{n+1} \ln x \, \mathrm{d}x - \int_1^n \ln x \, \mathrm{d}x}{\int_2^{n+1} \ln x \, \mathrm{d}x} = 0$$

Note that the numerator is simply

$$\int_n^{n+1} \ln x \, \mathrm{d}x - C \leq \ln(n+1)$$

where $C = \int_1^2 \ln x \, \mathrm{d}x$ is a positive constant. Similarly, the denominator is

$$\int_2^{n+1} \ln x \, \mathrm{d}x \geq (n-1)\ln 2$$

so we have

$$0 \leq \left| \frac{\int_2^{n+1} \ln x \, \mathrm{d}x - \int_1^n \ln x \, \mathrm{d}x}{\int_2^{n+1} \ln x \, \mathrm{d}x} \right| \leq \frac{\ln(n+1)}{(n-1)\ln 2}$$

for large enough n. By the Squeeze Theorem (note that the right hand size goes to zero), we have

$$\lim_{n\to\infty} \frac{\int_2^{n+1} \ln x \, \mathrm{d}x - \int_1^n \ln x \, \mathrm{d}x}{\int_2^{n+1} \ln x \, \mathrm{d}x} = 0$$

so

$$\lim_{n\to\infty} \frac{\int_1^n \ln x \, \mathrm{d}x}{\int_2^{n+1} \ln x \, \mathrm{d}x} = 1$$

and thus

$$\int_1^n \ln x \, \mathrm{d}x \sim \int_2^{n+1} \ln x \, \mathrm{d}x$$

as needed. ■

Question 2

Show that

$$\ln(n!) \sim n \ln n - n$$

Solution. We have

$$\begin{aligned} \int_1^n \ln x \,\mathrm{d}x &= \sum_{k=1}^{n-1} \int_k^{k+1} \ln x \,\mathrm{d}x \\ &\leq \sum_{k=1}^{n-1} \ln(k+1) \\ &= \ln\left(\prod_{k=2}^{n} k\right) = \ln(n!) \end{aligned}$$

and likewise

$$\begin{aligned} \int_2^{n+1} \ln x \,\mathrm{d}x &= \sum_{k=2}^{n} \int_k^{k+1} \ln x \,\mathrm{d}x \\ &\geq \sum_{k=2}^{n} \ln k \\ &= \ln\left(\prod_{k=2}^{n} k\right) = \ln(n!) \end{aligned}$$

so we have

$$\int_1^n \ln x \,\mathrm{d}x \leq \ln(n!) \leq \int_2^{n+1} \ln x \,\mathrm{d}x$$

dividing everything by $\int_2^{n+1} \ln x \,\mathrm{d}x$ and applying Question 1 and the Squeeze Theorem gives

$$\ln(n!) \sim \int_1^n \ln x \,\mathrm{d}x = n \ln n - n + 1 \sim n \ln n - n$$

as needed. ■

Question 3

Find the Taylor series of $\ln(1+x)$ centered at 0. Indicate the radius of convergence and show that the Taylor series converges to $\ln(1+x)$ for any $x \in (-1,1)$.

Solution.

Let $f(x) = \ln(1+x)$. We claim $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ for all $n \geq 1$. Indeed, we have in the base case

$$f^{(1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x) = (1+x)^{-1}$$

and if the claim holds for some $k \geq 1$, then

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(-1)^{k-1}(k-1)!(1+x)^{-k} = (-1)^k k!(1+x)^{-(k+1)}$$

so the Taylor series at zero is

$$\sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!(1+0)^{-n}\frac{x^n}{n!} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Note that we drop the $n = 0$ term since $f(0) = \ln(1) = 0$.

To find the radius of convergence, we use the ratio test. We have

$$\lim_{n\to\infty}\frac{\left|(-1)^n\frac{x^{n+1}}{n+1}\right|}{(-1)^{n-1}}\frac{x^n}{n}) = \lim_{n\to\infty}|x|\frac{n}{n+1} = |x|$$

so the series converges whenever $|x| < 1$ (i.e., the radius of convergence is 1).

Finally, we show that the Taylor series converges to $\ln(1+x)$ for any $x \in (-1,1)$. The $x = 0$ case is clear.

Suppose $x \in (0,1)$. Using the Lagrange remainder, we can write for all n,

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}x^{n+1}$$

for some $c_x \in (0,x)$. But,

$$f^{(n+1)}(c_x) = (-1)^n n!(1+c)^{-(n+1)}$$

where, since $c_x > 0$, $(1+c_x)^{-(n+1)} \leq 1$, so

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c_x)}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{n+1} \longrightarrow 0$$

and we have convergence to $f(x)$ in this case.

Suppose $x \in (-1,0)$. This time, we use the Cauchy remainder and write

$$\begin{aligned}
|R_n(x)| = \left|\frac{f^{(n+1)}(c_x)}{n!}(x-c_x)^n x\right| &= \frac{(c-x)^n}{(1+c)^n}\frac{|x|}{1+c} \\
&= \frac{|x|}{1+c}\left(1-\frac{1+x}{1+c}\right)^n \\
&< \frac{|x|}{1+c}(1-(1+x))^n && \text{since } 1+c<1, (1+x)/(1+c) > 1+x \\
&= \frac{|x|}{1+c}|x|^n && \text{note that } |x| = -x \text{ since } x < 0
\end{aligned}$$

As $|x| < 1$, this goes to zero as $n \longrightarrow \infty$ so we have convergence to $f(x)$ in this case too. ■

Question 4

Let

$$\ln(n!) = n\ln n - n + \frac{1}{2}\ln n + C_n$$

Show that $\lim_{n\to\infty} C_n$ exists.

Solution. By definition, we have

$$C_n = \ln(n!) - n\ln n + n - \frac{1}{2}\ln n$$

so

$$\begin{aligned} C_{n+1} - C_n &= \left(\ln((n+1)!) - (n+1)\ln(n+1) + (n+1) - \frac{1}{2}\ln(n+1)\right) - \left(\ln(n!) - n\ln n + n - \frac{1}{2}\ln n\right) \\ &= \ln(n+1) - (n+1)\ln(n+1) + 1 - \frac{1}{2}\ln(n+1) + n\ln n + \frac{1}{2}\ln n \\ &= 1 - n\ln\left(\frac{n+1}{n}\right) - \frac{1}{2}\ln\left(\frac{n+1}{n}\right) \\ &= 1 - \left(n + \frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) \end{aligned}$$

Now, $\ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right)$ so by Question 3, we have

$$\begin{aligned} C_{n+1} - C_n &= 1 - \left(n + \frac{1}{2}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{kn^k} \\ &= 1 - \left(n + \frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + o(n^{-3})\right) \\ &= -\frac{1}{12n^2} + o(n^{-2}) \end{aligned}$$

For sufficiently large n, this is negative so C_n is eventually decreasing. Moreover, C_n is eventually bounded below since it is easy to see $C_n > 0$ for large enough n. We conclude that C_n converges. ■

Question 5

Show that

$$\lim_{n\to\infty} C_n = \frac{1}{2}\ln 2\pi$$

Solution. By definition, $C_n = \ln(n!) - n\ln n + n - \frac{1}{2}\ln n$ from which we obtain

$$K_n = e^{C_n} = \frac{n!e^n}{n^{n+\frac{1}{2}}}$$

and thus

$$n! = e^{-n}n^{n+\frac{1}{2}}K_n$$

Now, by the Wallis formula, we have

$$\begin{aligned}
\frac{\pi}{2} &= \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n)!^2(2n+1)} \\
&= \lim_{n\to\infty} \frac{2^{4n}\left(e^{-n}n^{n+\frac{1}{2}}K_n\right)^4}{\left(e^{-2n}(2n)^{2n+\frac{1}{2}}K_{2n}\right)^2(2n+1)} \\
&= \lim_{n\to\infty} \frac{2^{4n}e^{-4n}n^{4n+2}K_n^4}{e^{-4n}(2n)^{4n+1}K_{2n}^2(2n+1)} \\
&= \lim_{n\to\infty} \frac{nK_n^4}{2K_{2n}^2(2n+1)} \\
&= \frac{1}{4}\lim_{n\to\infty} \frac{K_n^4}{K_{2n}^2} \\
&= \frac{1}{4}K^2 \qquad \text{where } K = \lim_{n\to\infty} K_n
\end{aligned}$$

In the last step, we take advantage of the fact that we know K_n converges and thus K_n and K_{2n} converge to the same thing (which must be non-zero, as $K_n = e^{C_n} > 0$).

From the above equation, we obtain $K = \sqrt{2\pi}$ and so $\lim_{n\to\infty} C_n = \ln(\lim_{n\to\infty} K_n) = \frac{1}{2}\ln 2\pi$. ■

Question 6

Prove the Stirling formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Solution.

Keeping the notation from Question 4, we have

$$\ln(n!) = n\ln n - n + \frac{1}{2}\ln n + C_n$$

where by Question 5, we have

$$C_n \sim \frac{1}{2}\ln 2\pi$$

so

$$\ln(n!) \sim n\ln n - n + \frac{1}{2}\ln n + \frac{1}{2}\ln 2\pi$$

Now, if $f \sim g$, then $\lim_{n\to\infty}(f(n) - g(n)) = 0$ so $e^f \sim e^g$. Thus, by exponentiating both sides, we get

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{n}\sqrt{2\pi}$$

which is Stirling's formula. ■