MAT159 Test Solutions – Test #1

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Question (A/B)1

Let $\mathcal{F}$ be the set of all functions $\mathbb{R} \to \mathbb{R}$ and $\mathcal{D} \subseteq \mathcal{F}$ be the set of all *differentiable* functions. Define the relation $\sim$ on $\mathcal{D}$ by

$$f \sim g := \exists c \in \mathbb{R}. \forall x \in \mathbb{R}. f(x) = g(x) + c$$

Show that $\sim$ is an equivalence relation on $\mathcal{D}$. Prove that the map $d : \mathcal{D}/_\sim \to \mathcal{F}$ given by $d([f]) = f'$ is well-defined and injective.

Solution. Fix $f, g, h \in \mathcal{D}$.

Then $f = f + 0$ so $f \sim f$.

If $f \sim g$, then $f = g + c$ for some $c \in \mathbb{R}$ so $g = f + (-c)$ and hence $g \sim f$.

If $f \sim g$ and $g \sim h$, then $f = g + c_1, g = h + c_2$ and hence $f = h + (c_1 + c_2)$ so $f \sim h$.

Thus, $\sim$ is an equivalence relation. Now, if $[f] = [g]$, then $f \sim g$ so $f = g + c$ and hence

$$\begin{aligned} d([f]) &= f' \\ &= (g + c)' \\ &= g' \\ &= d([g]) \end{aligned}$$

so d is a well-defined map on $\mathcal{D}/_\sim$.

Finally, suppose $d([f]) = d([g])$ which is to say $f' = g'$. Then, $(f - g)' = 0$ on all of $\mathbb{R}$ so by connectedness of $\mathbb{R}$, we have $f - g = c$ for some constant $c \in \mathbb{R}$. That is, $f(x) = g(x) + c$ for all $x \in \mathbb{R}$, so $f \sim g$ and hence $[f] = [g]$. Thus, d is injective. ∎

Remark. We can replace “set of all functions $\mathbb{R} \to \mathbb{R}$” with “set of all functions $C \to \mathbb{R}$” where $C \subseteq \mathbb{R}$ is any connected set. Without connectedness, the result fails.

Remark. This question is a fancy way of saying “antiderivatives are unique up to an additive constant”.

Question A2

Compute

$$\int \frac{1}{x(1+\ln x)(1+\ln(1+\ln x))}\,\mathrm{d}x$$

Solution. We first substitute $u = 1 + \ln x$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and thus

$$\int \frac{1}{x(1+\ln x)(1+\ln(1+\ln x))}\,\mathrm{d}x = \int \frac{1}{u(1+\ln u)}\,\mathrm{d}u$$

We substitute again $v = 1 + \ln u$ so that $\mathrm{d}v = \frac{1}{u}\,\mathrm{d}u$ and thus

$$\int \frac{1}{u(1+\ln u)}\,\mathrm{d}u = \int \frac{1}{v}\,\mathrm{d}v = \ln|v| + C$$

As $v = 1 + \ln u = 1 + \ln(1 + \ln x)$, we have

$$\int \frac{1}{x(1+\ln x)(1+\ln(1+\ln x))}\,\mathrm{d}x = \ln|1+\ln(1+\ln x)| + C$$

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Question B2

Compute

$$\int \frac{e^{\sqrt{1+x^2}}x}{\sqrt{1+x^2}}\,\mathrm{d}x$$

Solution. We first substitute $u = 1 + x^2$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$ so that $\frac{1}{2}\,\mathrm{d}u = x\,\mathrm{d}x$ and hence

$$\int \frac{e^{\sqrt{1+x^2}}x}{\sqrt{1+x^2}}\,\mathrm{d}x = \int \frac{e^{\sqrt{u}}}{2\sqrt{u}}\,\mathrm{d}u$$

Now substitute $v = \sqrt{u}$; we have $\mathrm{d}v = \frac{1}{2\sqrt{u}}\,\mathrm{d}u$ whence

$$\int \frac{e^{\sqrt{u}}}{2\sqrt{u}}\,\mathrm{d}u = \int e^v\,\mathrm{d}v = e^v + C$$

As $v = \sqrt{u} = \sqrt{1+x^2}$ we have

$$\int \frac{e^{\sqrt{1+x^2}}x}{\sqrt{1+x^2}}\,\mathrm{d}x = e^{\sqrt{1+x^2}} + C$$

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Remark. In both problems, we could of course have jumped directly to the final larger substitution ($v = 1 + \ln(1 + \ln x)$ in A2 and $v = \sqrt{1+x^2}$ in B2), but multiple small substitutions can sometimes be easier.

Question A3

Compute

$$\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,\mathrm{d}x$$

Solution. Substitute $u = \ln x$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and hence

$$\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = \arcsin(u) + C$$

As $u = \ln x$ we have

$$\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,\mathrm{d}x = \arcsin(\ln x) + C$$

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Question B3

Compute

$$\int \frac{1}{x(1+(\ln x)^2)}\,\mathrm{d}x$$

Substitute $u = \ln x$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and hence

$$\int \frac{1}{x(1+(\ln x)^2)}\,\mathrm{d}x = \int \frac{1}{1+u^2}\,\mathrm{d}u = \arctan(u) + C$$

As $u = \ln x$, we have

$$\int \frac{1}{x(1+(\ln x)^2)}\,\mathrm{d}x = \arctan(\ln x) + C$$

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