MAT159 Test Solutions – Test #2

Mustafa Motiwala

2026-01-22

Question (A/B)1

Let $I : \mathbb{R}[x] \to x\mathbb{R}[x]$ be a linear isomorphism satisfying $I(1) = x$ and

$$I(pI^{-1}(q)) = pq - I(qI^{-1}(p)) \qquad \forall p, q \in x\mathbb{R}[x]$$

State and prove an explicit formula for I.

Solution. We claim that $I(p) = \int p$ (where we choose the constant of integration to be 0). That is,

$$I(a_0 + a_1 x + \cdots + a_n x^n) = a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1}$$

As I is linear, it suffices to show $I(x^n) = \frac{1}{n+1} x^{n+1}$ for all $n \geq 0$. The $n = 0$ case is given by assumption, for we have $I(x^0) = I(1) = x$.

Now, suppose $I(x^n) = \frac{1}{n+1} x^{n+1}$ and consider $I(x^{n+1})$. Observe

$$\begin{aligned} I(x^{n+1}) &= I(x^{n+1} \cdot 1) \\ &= I(x^{n+1} I^{-1}(x)) \\ &= x^{n+2} - I(I^{-1}(x^{n+1}) \cdot x) \end{aligned}$$

From $I(x^n) = \frac{1}{n+1} x^{n+1}$ and linearity of I^{-1}, we obtain $\frac{1}{n+1} I^{-1}(x^{n+1}) = x^n$, so

$$\begin{aligned} &= x^{n+2} - I((n+1)x^{n+1}) \\ &= x^{n+2} - (n+1) I(x^{n+1}) \end{aligned}$$

Rearranging terms, we get

$$(n+2) I(x^{n+1}) = x^{n+2}$$

and thus

$$I(x^{n+1}) = \frac{1}{n+2} x^{n+2}$$

By induction, we have $I(x^n) = \frac{1}{n+1} x^{n+1}$ for all $n \geq 0$. Linearity of I gives the claimed formula. ∎

Remark. The given identity for I is just the integration by parts formula. We know I is the integral and thus by the FTC I^{-1} is the differential. So, the given identity becomes $\int p \, \mathrm{d}q = pq - \int q \, \mathrm{d}p$.

Follow-up. This problem shows that linearity + the IBP formula determines the integral on polynomials. Come up with (and of course prove) a similar type of statement for the derivative, e.g. something like "linearity + ____ determines the derivative on polynomials".

Question A2

Compute

$$\int e^{2t} \cos\left(\frac{1}{4}t\right) \mathrm{d}t$$

Solution. We use the technique of integration by parts. We need to choose one term to integrate and one to differentiate. In this problem, it turns out the choice doesn't matter (this means you should try it the other way and see what happens!); we'll choose

$$u = \cos\left(\frac{1}{4}t\right) \qquad \text{and} \qquad \mathrm{d}v = e^{2t}$$

so that

$$\mathrm{d}u = -\frac{1}{4}\sin\left(\frac{1}{4}t\right) \qquad \text{and} \qquad v = \frac{1}{2}e^{2t}$$

Then,

$$\int e^{2t} \cos\left(\frac{1}{4}t\right) \mathrm{d}t = \frac{1}{2}e^{2t}\cos\left(\frac{1}{4}t\right) + \frac{1}{8}\int e^{2t}\sin\left(\frac{1}{4}t\right) \mathrm{d}t$$

We again integrate by parts, and again choose to integrate the exponential and differentiate the sine, so

$$u = \sin\left(\frac{1}{4}t\right) \qquad \text{and} \qquad \mathrm{d}v = e^{2t}$$

so

$$\mathrm{d}u = \frac{1}{4}\cos\left(\frac{1}{4}t\right) \qquad \text{and} \qquad v = \frac{1}{2}e^{2t}$$

which gives

$$\int e^{2t} \sin\left(\frac{1}{4}t\right) \mathrm{d}t = \frac{1}{2}e^{2t}\sin\left(\frac{1}{4}t\right) - \frac{1}{8}\int e^{2t}\cos\left(\frac{1}{4}t\right) \mathrm{d}t$$

Notice that the remaining integral is the same as what we started with. Thus, from

$$\int e^{2t} \cos\left(\frac{1}{4}t\right) \mathrm{d}t = \frac{1}{2}e^{2t}\cos\left(\frac{1}{4}t\right) + \frac{1}{8}\left(\frac{1}{2}e^{2t}\sin\left(\frac{1}{4}t\right) - \frac{1}{8}\int e^{2t}\cos\left(\frac{1}{4}t\right) \mathrm{d}t\right)$$

we obtain

$$\frac{65}{64}\int e^{2t} \cos\left(\frac{1}{4}t\right) \mathrm{d}t = \frac{1}{2}e^{2t}\cos\left(\frac{1}{4}t\right) + \frac{1}{16}e^{2t}\sin\left(\frac{1}{4}t\right)$$

and thus

$$\int e^{2t} \cos\left(\frac{1}{4}t\right) \mathrm{d}t = \frac{1}{65}e^{2t}\left(32\cos\left(\frac{1}{4}t\right) + 4\sin\left(\frac{1}{4}t\right)\right) + C$$

■

Question B2

Compute

$$\int (3t + t^2) \sin(2t)\, \mathrm{d}t$$

Solution. We use the technique of integration by parts. We need to choose one factor to integrate and another to differentiate.

As integrating a polynomial only results in a more complicated polynomial, it's probably better to differentiate that factor and integrate the sine term. Thus, we set

$$u = 3t + t^2 \qquad \text{and} \qquad \mathrm{d}v = \sin(2t)$$

so that

$$\mathrm{d}u = 2t + 3 \qquad \text{and} \qquad v = -\frac{1}{2}\cos(2t)$$

Then,

$$\int (3t + t^2) \sin(2t)\, \mathrm{d}t = -\frac{1}{2}(3t + t^2)\cos(2t) + \frac{1}{2}\int (2t + 3)\cos(2t)\, \mathrm{d}t$$

For the remaining integral we integrate by parts again, setting

$$u = 2t + 3 \qquad \text{and} \qquad \mathrm{d}v = \cos(2t)$$

so that

$$\mathrm{d}u = 2 \qquad \text{and} \qquad v = \frac{1}{2}\sin(2t)$$

Then,

$$\begin{aligned}\int (2t + 3)\cos(2t)\, \mathrm{d}t &= \frac{1}{2}(2t + 3)\sin(2t) - \int \sin(2t)\, \mathrm{d}t \\ &= \frac{1}{2}(2t + 3)\sin(2t) + \frac{1}{2}\cos(2t)\end{aligned}$$

and so

$$\begin{aligned}\int (3t + t^2) \sin(2t)\, \mathrm{d}t &= -\frac{1}{2}(3t + t^2)\cos(2t) + \frac{1}{2}\int (2t + 3)\cos(2t)\, \mathrm{d}t \\ &= -\frac{1}{2}(3t + t^2)\cos(2t) + \frac{1}{4}(2t + 3)\sin(2t) + \frac{1}{4}\cos(2t) + C\end{aligned}$$

■

Question A3

Compute

$$\int \frac{8}{3t^3 + 7t^2 + 4t} \, \mathrm{d}t$$

Solution. We need to find a partial fraction decomposition. First, we factor the denominator

$$3t^3 + 7t^2 + 4t = t(3t + 4)(t + 1)$$

and aim to find $A, B, C \in \mathbb{R}$ such that

$$\frac{8}{3t^3 + 7t^2 + 4t} = \frac{A}{t} + \frac{B}{3t + 4} + \frac{C}{t + 1}$$

Multiplying through we get

$$8 = A(3t + 4)(t + 1) + Bt(t + 1) + Ct(3t + 4)$$

This equation is true for *all* t so by choosing specific values of t to make all but one of the terms on the right-hand side vanish, we can easily solve for A, B, C.

For example, when $t = 0$, the equation becomes

$$8 = 4A \Longrightarrow A = 2$$

Likewise, setting $t = -\frac{4}{3}$ yields

$$8 = \frac{4}{9}B \Longrightarrow B = 18$$

and from $t = -1$ we get

$$8 = -C \Longrightarrow C = -8$$

Thus,

$$\begin{aligned}
\int \frac{8}{3t^3 + 7t^2 + 4t} \, \mathrm{d}t &= \int \left(\frac{2}{t} + \frac{18}{3t + 4} - \frac{8}{t + 1} \right) \mathrm{d}t \\
&= \int \frac{2}{t} \, \mathrm{d}t + \int \frac{18}{3t + 4} \, \mathrm{d}t - \int \frac{8}{t + 1} \, \mathrm{d}t \\
&= 2 \ln|t| + 6 \ln|3t + 4| - 8 \ln|t + 1| + C
\end{aligned}$$

■

Question B3

Compute

$$\int \frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}\,\mathrm{d}t$$

Solution. We need to find a partial fraction decomposition. The denominator is already factored, so we need to find $A, B, C, D \in \mathbb{R}$ such that

$$\frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)} = \frac{At+B}{3t^2+4} + \frac{Ct+D}{t^2+7}$$

Clearing denominators we get the polynomial equation in t

$$\begin{aligned} 8+t+6t^2-12t^3 &= (At+B)(t^2+7)+(Ct+D)(3t^2+4) \\ &= (A+3C)t^3+(B+3D)t^2+(7A+4C)t+(7B+4D) \end{aligned}$$

Equating coefficients, we obtain the linear system

$$\begin{aligned} t^0 :&\quad 8 = 7B+4D \\ t^2 :&\quad 6 = B+3D \\ \\ t^1 :&\quad 1 = 7A+4C \\ t^3 :&\quad -12 = A+3C \end{aligned}$$

which are really two systems

$$\begin{pmatrix} 7 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 7 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -12 \end{pmatrix}$$

Since

$$\begin{pmatrix} 7 & 4 \\ 1 & 3 \end{pmatrix}^{-1} = \frac{1}{17}\begin{pmatrix} 3 & -4 \\ -1 & 7 \end{pmatrix}$$

we get

$$\begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

Thus,

$$\begin{aligned} \int \frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}\,\mathrm{d}t &= \int \left(\frac{3t}{3t^2+4} + \frac{-5t+2}{t^2+7}\right)\mathrm{d}t \\ &= \int \frac{3t}{3t^2+4}\,\mathrm{d}t - \int \frac{5t}{t^2+7}\,\mathrm{d}t + \int \frac{2}{t^2+7}\,\mathrm{d}t \end{aligned}$$

For the first and second integrals, we make the substitutions $u = 3t^2+4$ and $u = t^2+7$ respectively and get

$$= \frac{1}{2}\ln|3t^2+4| - \frac{5}{2}\ln|t^2+7| + \int \frac{2}{t^2+7}\,\mathrm{d}t$$

For the last integral, make the substitution $u = t/\sqrt{7}$ and obtain

$$= \frac{1}{2}\ln|3t^2+4| - \frac{5}{2}\ln|t^2+7| + \frac{2}{\sqrt{7}}\arctan\left(\frac{t}{\sqrt{7}}\right) + C$$

■