MAT159 Test Solutions – Test #3

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Question (A/B)1

Let $S = \bigcup_{k=1}^{n}$ be a *disjoint* union of intervals $I_k = [a_k, b_k]$. Let χ_S be the characteristic function of S. Show that for any $S \subseteq [a, b]$, we have

$$\int_a^b \chi_S = \sum_{k=1}^{n} (b_k - a_k)$$

Solution. We first show the result for $n = 1$; i.e, when $S = [a_1, b_1]$ is an interval. Fix some $[a, b] \supseteq [a_1, b_1]$. Let $\varepsilon > 0$, choose $\delta = \varepsilon/2$, and let (Γ, η) be a marked partition of $[a, b]$ with $\|\Gamma\| < \delta$. Write $\Gamma = x_0 < x_1 < \dots < x_n$ and choose $0 \leq i < j < n$ to be the smallest and largest indices respectively such that $\eta_i, \eta_j \in [a_1, b_1]$. That is, $\eta_k \in [a_1, b_1]$ if and only if $i \leq k \leq j$. Then, we have

$$\begin{aligned} \sigma(\chi_S, \Gamma, \eta) &= \sum_{k=0}^{n-1} \chi_S(\eta_k)(x_{k+1} - x_k) \\ &= \sum_{k=i}^{j} (x_{k+1} - x_k) \\ &= x_{j+1} - x_i \end{aligned}$$

From $\|\Gamma\| < \delta = \varepsilon/2$ and the fact that $\eta_j \in [a_1, b_1]$ but $\eta_{j+1} \notin [a_1, b_1]$, we have

$$x_{j+1} - \varepsilon/2 \leq \eta_j \leq b_1 < \eta_{j+1} \leq x_{j+1} + \varepsilon/2$$

and thus $|x_{j+1} - b| < \varepsilon/2$. Similarly, we have $|x_i - a_1| < \varepsilon/2$. So, writing $x_{j+1} = b_1 + s$ and $x_i = a_1 + t$ with $|s|, |t| < \varepsilon/2$, we have

$$\begin{aligned} |\sigma(\chi_S, \gamma, \eta) - (b_1 - a_1)| &= \left|x_{j+1} - x_i - (b_1 - a_1)\right| \\ &= |(b_1 + s) - (a_1 + t) - (b_1 - a_1)| \\ &= |s - t| \\ &\leq |s| + |t| < \varepsilon \end{aligned}$$

as needed. Thus, $\int_a^b \chi_S = b_1 - a_1$ when $S = [a_1, b_1]$.

To extend to the general case $S = \bigcup_{k=1}^{n} I_k$, observe that when the I_k are disjoint, we have

$$\chi_S = \sum_{k=1}^{n} \chi_{I_k}$$

and thus by linearity of the Riemann integral, we have

$$\begin{aligned} \int_a^b \chi_S &= \sum_{k=1}^{n} \int_a^b \chi_{I_k} \\ &= \sum_{k=1}^{n} |b_k - a_k| \qquad \text{by the result just proved} \end{aligned}$$

which was to be shown. ∎

Remark. This problem shows that (for one specific type of set S) integrating the characteristic of S gives the "size" or *measure* of S.

This turns out to be true in general, i.e for any "measurable" set S, we have $\int_{-\infty}^{\infty} \chi_S = \mu(S)$, where $\mu(S)$ is the measure of S. This holds in higher dimensions too; for example, if S is the unit square, then $\int_{\mathbb{R}^2} \chi_S = 1$ and if S is the unit circle, then $\int_{\mathbb{R}^2} \chi_S = \pi$. Let us define higher dimensional integrals as simply iterated one dimensional integrals, so that $\int_{\mathbb{R}^2} \chi_S = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \chi_S(x, y) \, \mathrm{d}x \right) \mathrm{d}y$.

Bonus problem. Inspired by this problem and the above remark, we define $\mu(S)$ for (some) subsets $S \subseteq \mathbb{R}^n$ as follows: if $S \subseteq [a_1, b_1] \times \cdots \times [a_n, b_n]$, then

$$\mu(S) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \chi_S(x_1, ..., x_n) \, \mathrm{d}x_n \cdots \mathrm{d}x_2 \, \mathrm{d}x_1$$

Compute, for all $n \in \mathbb{N}$, $\mu(\Delta_n)$ where Δ_n is the n-dimensional pyramid

$$\Delta_n := \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1, x_2, ..., x_n \geq 0, x_1 + x_2 + \cdots + x_n \leq 1\}$$

Question A2

Let $f : [a,b] \to \mathbb{R}$ be bounded with $f \in \mathfrak{R}[t,b]$ for all $t \in (a,b)$. Prove that $f \in \mathfrak{R}[a,b]$ and $\int_a^b f = \lim_{t\to a^+} \int_t^b f$.

Solution. There are two ways to approach the problem: first show the integral exists and then show the limit is equal to it, or first show the limit exists and then show the integral is equal to it. We present both solutions.

Integral first

As f is bounded, choose $M > 0$ with $f \leq M$. We first show $f \in \mathfrak{R}[a,b]$ by applying the Cauchy criterion.

Let $\varepsilon > 0$. Set $t = a + \frac{\varepsilon}{8M}$ and choose by integrability of f over $[t,b]$ some $\delta > 0$ such that both $\delta < \frac{\varepsilon}{8M}$ and for any marked partitions $(\Gamma_1, \eta_1), (\Gamma_2, \eta_2)$ of $[t,b]$ with $\|\Gamma_1\|, \|\Gamma_2\| < \delta$ we have $|\sigma(f,\Gamma_1,\eta_1) - \sigma(f,\Gamma_2,\eta_2)| < \varepsilon/4$.

Fix partitions $(\Gamma_1, \eta_1), (\Gamma_2, \eta_2)$ of $[a,b]$. Let $(\Gamma_1', \eta_1'), (\Gamma_2', \eta_2')$ be the partitions obtained from these by adding t (to both Γ and η) if necessary. This can not increase the size, so we have $\|\Gamma_1'\|, \|\Gamma_2'\| < \delta$. Then we can write

$$\Gamma_1' = \underbrace{x_0 < \cdots <}_{\Gamma'_{1,<t}} \underbrace{t < \cdots < x_n}_{\Gamma'_{1,>t}}$$
$$\Gamma_2' = \underbrace{\tilde{x}_0 < \cdots <}_{\Gamma'_{2,<t}} \underbrace{t < \cdots < \tilde{x}_m}_{\Gamma'_{2,>t}}$$

so that

$$\begin{aligned} |\sigma(f,\Gamma_1',\eta_1') - \sigma(f,\Gamma_2',\eta_2')| &\leq \left|\sigma\left(f,\Gamma'_{1,<t},\eta'_{1,<t}\right) - \sigma\left(f,\Gamma'_{2,<t},\eta'_{2,<t}\right)\right| + \left|\sigma\left(f,\Gamma'_{1,>t},\eta'_{1,>t}\right) - \sigma\left(f,\Gamma'_{2,>t},\eta'_{2,>t}\right)\right| \\ &< 2M(t-a) + \varepsilon/4 = \frac{\varepsilon}{2} \end{aligned}$$

Since Γ_1', Γ_2' are equal to Γ_1, Γ_2 except in perhaps two subintervals, we have

$$\begin{aligned} |\sigma(f,\Gamma_1,\eta_1) - \sigma(f,\Gamma_2,\eta_2)| &\leq |\sigma(f,\Gamma_1',\eta_1') - \sigma(f,\Gamma_2',\eta_2')| + 4M\delta \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, by the Cauchy criterion for Riemann integrability, the integral $\int_a^b f$ exists. It is then easy to see that $\lim_{t\to a^+} \int_t^b f = \int_a^b f$ for given $\varepsilon > 0$ we can choose $\delta = \frac{\varepsilon}{M}$ so that if $t \in (a, a+\delta)$ we have

$$\left|\int_t^b f - \int_a^b f\right| \leq \int_a^t |f| < M\delta = \varepsilon$$

as needed. ■

Limit first

We first show the existence of the limit by using Cauchy's criterion for continuous limits which is stated in the following lemma.

Lemma. *For g a function and $c \in \mathbb{R}$, we have $\lim_{x\to c} g(x)$ exists if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in U_\delta(c)$ we have $|g(x) - g(y)| < \varepsilon$.*

Proof. The ($\Longrightarrow$) direction is a trivial application of the triangle inequality. For the other direction, assume the Cauchy criterion. Our assumption implies the sequence $g(c_n)$ for $c_n = c + 1/n$ is Cauchy and hence converges to a real number $\alpha \in \mathbb{R}$. We claim $\lim_{x\to c} g(x) = \alpha$. Let $\varepsilon > 0$ and choose $\delta > 0$ so that $x, y \in U_\delta(c)$ implies $|g(x) - g(y)| < \varepsilon/2$. Fix x with $0 < |x - c| < \delta$. Choose N sufficiently large so that $\frac{1}{N} < \delta$ and $|g(c_N) - \alpha| < \varepsilon/2$; then,

$$|g(x) - \alpha| \leq |g(x) - g(c_N)| + |g(c_N) - \alpha| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which concludes the proof. ■

We apply this lemma to the limit in question. As f is bounded, choose $M > 0$ with $|f| \leq M$. Let $\varepsilon > 0$. Take $\delta = \varepsilon/M$. If $t_1, t_2 \in (a, a+\delta)$, then $|t_1 - t_2| < \delta$ so

$$\begin{aligned}\left|\int_{t_1}^{b} f - \int_{t_2}^{b} f\right| &= \left|\int_{t_1}^{t_2} f\right| \\ &\leq \int_{t_1}^{t_2} |f| \\ &< M\delta = \varepsilon\end{aligned}$$

Thus, the limit $\lim_{t\to a^+} \int_t^b f$ exists, so say $S \coloneqq \lim_{t\to a^+} \int_t^b f$.

We claim $\int_a^b f = S$. Let $\varepsilon > 0$ be given. Set $t = a + \frac{\varepsilon}{4M}$. Choose $\delta > 0$ so that $\delta < \frac{\varepsilon}{8M}$, $\left|\int_{t'}^b f - S\right| < \frac{\varepsilon}{4}$ whenever $t' \in (a, a+\delta)$, and $\left|\sigma(f,\Gamma,\eta) - \int_t^b f\right| < \frac{\varepsilon}{4}$ whenever $\|\Gamma\| < \delta$

Let (Γ, η) be a marked partition with $\|\Gamma\| < \delta$. Insert t into Γ, η if necessary to obtain (Γ', η') which splits into $(\Gamma'_{<t}, \eta'_{<t})$ and $(\Gamma'_{>t}, \eta'_{>t})$. We have then that $\|\Gamma'\| < \delta$ and

$$|\sigma(f,\Gamma,\eta) - S| \leq |\sigma(f,\Gamma,\eta) - \sigma(f,\Gamma',\eta')| + |\sigma(f,\Gamma',\eta') - S|$$

As Γ, Γ' differ in only two subintervals, most terms in the first summand cancel out so that

$$\begin{aligned}&\leq 2M\delta + |\sigma(f,\Gamma',\eta') - S| \\ &\leq 2M\delta + |\sigma(f,\Gamma'_{<t},\eta'_{<t})| + \left|\sigma(f,\Gamma'_{>t},\eta'_{>t}) - \int_t^b f\right| + \left|\int_t^b f - S\right| \\ &\leq 2M\delta + M(t-a) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon\end{aligned}$$

which shows $f \in \mathfrak{R}[a,b]$ with $\int_a^b f = \lim_{t\to a^+} \int_t^b f$. ■

Question B2

Let $f : [a, b] \to \mathbb{R}$ be bounded with $f \in \mathfrak{R}[a, t]$ for all $t \in (a, b)$. Prove that $f \in \mathfrak{R}[a, b]$ and $\int_a^b f = \lim_{t \to b^-} \int_a^t f$.

Solution.

There are two ways to approach the problem: first show the integral exists and then show the limit is equal to it, or first show the limit exists and then show the integral is equal to it. We present both solutions. Both solutions are nearly identical to that presented above for A2.

Integral first

As f is bounded, choose $M > 0$ with $f \le M$. We first show $f \in \mathfrak{R}[a, b]$ by applying the Cauchy criterion.

Let $\varepsilon > 0$. Set $t = b - \frac{\varepsilon}{8M}$ and choose by integrability of f over $[a, t]$ some $\delta > 0$ such that both $\delta < \frac{\varepsilon}{4M}$ and for any marked partitions $(\Gamma_1, \eta_1), (\Gamma_2, \eta_2)$ of $[a, t]$ with $\|\Gamma_1\|, \|\Gamma_2\| < \delta$ we have $|\sigma(f, \Gamma_1, \eta_1) - \sigma(f, \Gamma_2, \eta_2)| < \varepsilon/4$.

Fix partitions $(\Gamma_1, \eta_1), (\Gamma_2, \eta_2)$ of $[a, b]$. Let $(\Gamma_1', \eta_1'), (\Gamma_2', \eta_2')$ be the partitions obtained from these by adding t (to both Γ and η) if necessary. This can not increase the size, so we have $\|\Gamma_1'\|, \|\Gamma_2'\| < \delta$. Then we can write

$$\Gamma_1' = \underbrace{x_0 < \cdots <}_{\Gamma_{1,<t}'} \underbrace{t < \cdots < x_n}_{\Gamma_{1,>t}'}$$
$$\Gamma_2' = \underbrace{\tilde{x}_0 < \cdots <}_{\Gamma_{2,<t}'} \underbrace{t < \cdots < \tilde{x}_m}_{\Gamma_{2,>t}'}$$

so that

$$\begin{aligned} |\sigma(f, \Gamma_1', \eta_1') - \sigma(f, \Gamma_2', \eta_2')| &\le \left|\sigma(f, \Gamma_{1,<t}', \eta_{1,<t}') - \sigma(f, \Gamma_{2,<t}', \eta_{2,<t}')\right| + \left|\sigma(f, \Gamma_{1,>t}', \eta_{1,>t}') - \sigma(f, \Gamma_{2,>t}', \eta_{2,>t}')\right| \\ &< \varepsilon/4 + 2M(b - t) = \frac{\varepsilon}{2} \end{aligned}$$

Since Γ_1', Γ_2' are equal to Γ_1, Γ_2 except in perhaps two subintervals, we have

$$\begin{aligned} |\sigma(f, \Gamma_1, \eta_1) - \sigma(f, \Gamma_2, \eta_2)| &\le |\sigma(f, \Gamma_1', \eta_1') - \sigma(f, \Gamma_2', \eta_2')| + 2M\delta \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, by the Cauchy criterion for Riemann integrability, the integral $\int_a^b f$ exists. It is then easy to see that $\lim_{t \to b^-} \int_a^t f = \int_a^b f$ for given $\varepsilon > 0$ we can choose $\delta = \frac{\varepsilon}{M}$ so that if $t \in (b - \delta, b)$ we have

$$\left| \int_a^t f - \int_a^b f \right| \le \int_t^b |f| < M\delta = \varepsilon$$

as needed. ■

Limit first

We first show the existence of the limit by using Cauchy's criterion for continuous limits which is stated in the following lemma.

Lemma. *For g a function and $c \in \mathbb{R}$, we have $\lim_{x \to c} g(x)$ exists if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in U_\delta(c)$ we have $|g(x) - g(y)| < \varepsilon$.*

Proof. The $(\Longrightarrow)$ direction is a trivial application of the triangle inequality. For the other direction, assume the Cauchy criterion. Our assumption implies the sequence $g(c_n)$ for $c_n = c + 1/n$ is Cauchy and hence converges to a real number $\alpha \in \mathbb{R}$. We claim $\lim_{x \to c} g(x) = \alpha$. Let $\varepsilon > 0$ and choose $\delta > 0$ so that $x, y \in U_\delta(c)$ implies $|g(x) - g(y)| < \varepsilon/2$. Fix x with $0 < |x - c| < \delta$. Choose N sufficiently large so that $\frac{1}{N} < \delta$ and $|g(c_N) - \alpha| < \varepsilon/2$; then,

$$|g(x) - \alpha| \le |g(x) - g(c_N)| + |g(c_N) - \alpha| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which concludes the proof. ■

We apply this lemma to the limit in question. As f is bounded, choose $M > 0$ with $|f| \le M$. Let $\varepsilon > 0$. Take $\delta = \varepsilon/M$. If $t_1, t_2 \in (b - \delta, b)$, then $|t_1 - t_2| < \delta$ so

$$\begin{aligned}\left|\int_a^{t_1} f - \int_a^{t_2} f\right| &= \left|\int_{t_1}^{t_2} f\right| \\ &\le \int_{t_1}^{t_2} |f| \\ &< M\delta = \varepsilon\end{aligned}$$

Thus, the limit $\lim_{t\to b^-} \int_a^t f$ exists, so say $S := \lim_{t\to b^-} \int_a^t f$.

We claim $\int_a^b f = S$. Let $\varepsilon > 0$ be given. Set $t = b - \frac{\varepsilon}{4M}$. Choose $\delta > 0$ so that $\delta < \frac{\varepsilon}{8M}$, $\left|\int_a^{t'} f - S\right| < \frac{\varepsilon}{4}$ whenever $t' \in (b-\delta, b)$, and $\left|\sigma(f, \Gamma, \eta) - \int_a^t f\right| < \frac{\varepsilon}{4}$ whenever $\|\Gamma\| < \delta$

Let (Γ, η) be a marked partition with $\|\Gamma\| < \delta$. Insert t into Γ, η if necessary to obtain (Γ', η') which splits into $(\Gamma'_{<t}, \eta'_{<t})$ and $(\Gamma'_{>t}, \eta'_{>t})$. We have then that $\|\Gamma'\| < \delta$ and

$$|\sigma(f,\Gamma,\eta) - S| \le |\sigma(f,\Gamma,\eta) - \sigma(f,\Gamma',\eta')| + |\sigma(f,\Gamma',\eta') - S|$$

As Γ, Γ' differ in only two subintervals, most terms in the first summand cancel out so that

$$\begin{aligned}&\le 2M\delta + |\sigma(f,\Gamma',\eta') - S| \\ &\le 2M\delta + |\sigma(f,\Gamma'_{>t},\eta'_{>t})| + \left|\sigma(f,\Gamma'_{<t},\eta'_{<t}) - \int_a^t f\right| + \left|\int_a^t f - S\right| \\ &\le 2M\delta + M(b-t) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon\end{aligned}$$

which shows $f \in \mathfrak{R}[a,b]$ with $\int_a^b f = \lim_{t\to b^-} \int_a^t f$. ■

Question A3

Compute

$$\int t^3(3t^2-4)^{\frac{5}{2}}\,\mathrm{d}t$$

Solution.

We first set $t=\frac{2}{\sqrt{3}}s$ so that $\mathrm{d}t=\frac{2}{\sqrt{3}}\,\mathrm{d}s$ and thus

$$\begin{aligned}\int t^3(3t^2-4)\,\mathrm{d}t &= \int \frac{16}{9}s^3(4s^2-4)^{\frac{5}{2}}\,\mathrm{d}s\\ &= \int \frac{512}{9}s^3(s^2-1)^{\frac{5}{2}}\,\mathrm{d}s\end{aligned}$$

The integrand is in $\mathbb{R}\left(s,\sqrt{s^2-1}\right)$, so we make the substitution $s=\sec\theta$, $\mathrm{d}s=\sec\theta\tan\theta\,\mathrm{d}\theta$ and recall the identity $\sec^2\theta-1=\tan^2\theta$ to obtain

$$=\frac{512}{9}\int \sec^4\theta\tan^6\theta\,\mathrm{d}\theta$$

Now, as $\sec^2\theta=\tan^2\theta+1$ and $\mathrm{d}(\tan\theta)=\sec^2\theta$, the substitution $u=\tan\theta$ yields

$$\begin{aligned}&=\frac{512}{9}\int (u^2+1)u^6\,\mathrm{d}u\\ &=\frac{512}{9}\left(\frac{1}{9}u^9+\frac{1}{7}u^7\right)+C\\ &=\frac{512}{9}\left(\frac{1}{9}\tan^9\theta+\frac{1}{7}\tan^7\theta\right)+C\end{aligned}$$

We want to get this back into a function of t, so we first substitute $\tan\theta=\sec^2\theta-1=\sqrt{s^2-1}$ to get

$$=\frac{512}{9}\left(\frac{1}{9}(s^2-1)^{\frac{9}{2}}+\frac{1}{7}(s^2-1)^{\frac{7}{2}}\right)+C$$

Finally, we set $s=\frac{\sqrt{3}}{2}t$ so that

$$\begin{aligned}&=\frac{512}{9}\left(\frac{1}{9}\left(\frac{3}{4}t^2-1\right)^{\frac{9}{2}}+\frac{1}{7}\left(\frac{3}{4}t^2-1\right)^{\frac{7}{2}}\right)+C\\ &=\frac{(3t^2-4)^{\frac{9}{2}}}{81}+\frac{4(3t^2-4)^{\frac{7}{2}}}{63}+C\end{aligned}$$

Question B3

Compute

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}} \, \mathrm{d}x$$

Solution. We first complete the square to get

$$\begin{aligned} 9x^2 - 36x + 37 &= 9\left(x^2 - 4x + \frac{37}{9}\right) \\ &= 9\left(x^2 - 4x + 4 + \left(\frac{37}{9} - 4\right)\right) \\ &= 9\left((x-2)^2 + \frac{1}{9}\right) = 9(x-2)^2 + 1 \end{aligned}$$

so the integral becomes

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}} \, \mathrm{d}x = \int \frac{1}{\sqrt{9(x-2)^2 + 1}} \, \mathrm{d}x$$

We set $x = \frac{t}{3} + 2$ so that $\mathrm{d}x = \frac{1}{3} \, \mathrm{d}t$ and we get

$$\int \frac{1}{\sqrt{9(x-2)^2 + 1}} \, \mathrm{d}x = \frac{1}{3} \int \frac{1}{\sqrt{t^2 + 1}} \, \mathrm{d}t$$

The integrand is now in $\mathbb{R}\left(t, \sqrt{t^2+1}\right)$ so we substitute $t = \tan\theta$ with $\mathrm{d}t = \sec^2\theta \, \mathrm{d}\theta$ and obtain, recalling the identity $\tan^2\theta + 1 = \sec^2\theta$ the integral

$$\begin{aligned} &= \frac{1}{3} \int \sec\theta \, \mathrm{d}\theta \\ &= \frac{1}{3} \ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{3} \ln\left|\sqrt{t^2+1} + t\right| + C \\ &= \frac{1}{3} \ln\left|\sqrt{9(x-2)^2 + 1} + 3(x-2)\right| + C \end{aligned}$$