MAT159 Test Solutions – Test #4

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2026-02-06

Question 1

Let f be non-negative on $[a, b]$. Show that $\int_a^b f = 0$ implies $f = 0$ almost everywhere. Prove/disprove the converse.

Solution. Assume $\int_a^b f = 0$. Since f is non-negative, we have $f(x) \neq 0$ if and only if $f(x) > 0$, so we want to show that the set

$$E = \{x : f(x) > 0\}$$

is a null set. Since countable unions of null sets are null and

$$E = \bigcup_{n \geq 1} \underbrace{\left\{x : f(x) > \frac{1}{n}\right\}}_{E_n}$$

it suffices to show that each E_n is null.

So, fix $n \geq 1$ and $\varepsilon > 0$. Choose a partition $\Gamma = \{x_0, ..., x_m\}$ of $[a, b]$ with $\left|\overline{S}(f, \Gamma) - 0\right| < \varepsilon / n$. Take

$$U = \left\{[x_i, x_{i+1}] : 0 \leq i < m, M_i > \frac{1}{n}\right\}$$

Then, U is a cover of E_n since if $f(x) > 1/n$, then the sup of f over the subinterval containing x must be $> 1/n$. Enumerate $U = \left\{\left[x_{i_j}, x_{i_j+1}\right]\right\}_{j=1}^{k}$ and estimate

$$\begin{aligned} \sum_{j=1}^{k}\left(x_{i_j+1} - x_{i_j}\right) &= n \sum_{j=1}^{k} \frac{1}{n}\left(x_{i_j+1} - x_{i_j}\right) \\ &< n \sum_{i=1}^{m} M_i (x_{i+1} - x_i) && \text{since } f \text{ is non-negative} \\ &= n \overline{S}(f, \Gamma) \\ &< \varepsilon \end{aligned}$$

So, each E_n is a null set and thus $E = \bigcup_{n \geq 1} E_n$ is null. ■

The truth of the converse depends on the integrability of f.

If f is not assumed to be integrable, then the converse is not true. Let $f = \chi_Q$ be the Dirichlet function. Then f is non-negative and zero almost everywhere but χ_Q is not integrable over any interval.

However, the claim "if f is integrable and zero almost everywhere, then $\int_a^b f = 0$" is true.

Question 2

Let f, g be continuous over $[a, b]$ and $f \geq g$. Show that $f = g \iff \int_a^b f = \int_a^b g$.

Solution. The ($\implies$) direction is trivial. For the other direction, we first need a small lemma.

Lemma. *Let $h : [a, b] \to \mathbb{R}$ be continuous. If $h = 0$ almost everywhere, then $h = 0$.*

Proof. We show the contrapositive. Suppose $h \neq 0$ so that there is some $x_0 \in [a, b]$ for which $h(x_0) \neq 0$. Applying continuity of h at x_0 with $\varepsilon = |h(x_0)|/2$ we obtain some $\delta > 0$ such that $h(x) \neq 0$ for all $x \in U_\delta(x_0)$. Since $U_\delta(x_0)$ is not a null set, we see that h is not 0 almost everywhere. ■

Assume $\int_a^b f = \int_a^b g$ which implies $\int_a^b (f - g) = 0$. Since $f \geq g$, we have $f - g$ is a non-negative function. By Question 1, we thus have $f - g = 0$ almost everywhere. But, $f - g$ is continuous (since f, g are continuous), so by our lemma, $f - g = 0$ almost everywhere implies $f - g = 0$. That is, $f = g$. ■

Bonus problem. Define $\sim$ on $\mathfrak{R}[a, b]$ by

$$f \sim g := \text{“}f = g \text{ almost everywhere ”}$$

Show that $\sim$ is an equivalence relation.

True/False: For any $f, g \in \mathfrak{R}[a, b]$: $f \sim g \iff \int_a^b f = \int_a^b g$. If false, which implications are true? Can you add assumptions on f, g to make the statement true?

Question 3

Let $f : [a, b] \to \mathbb{R}$ be bounded. Show that if f is continuous almost everywhere on $[a, b]$, then f is Riemann integrable over $[a, b]$.

Proof. Assume f is continuous almost everywhere. Let $D_f = \{x \in [a, b] : f \text{ is not continuous at } x\}$. Let $M > 0$ be an upperbound for $|f|$. We will use the Cauchy criterion for Riemann integrability.

Fix $\varepsilon > 0$. Let $U = \{(\alpha_i, \beta_i)\}_{i \in I}$ be a countable open cover of D_f with total length $|U| = \sum_{i \in I}(\beta_i - \alpha_i) < \varepsilon/4M$.

For each $x \in [a, b] \setminus U$, we choose by continuity of f at $x \notin U$ some $\delta_x > 0$ such that for any $p, q \in U_{\delta_x}(x)$ we have $|f(p) - f(q)| < \varepsilon/4(b - a)$. The collection of all $U_{\delta_x/2}(x)$ alongside U forms an open cover of $[a, b]$ so by compactness of $[a, b]$ we obtain a finite subcover

$$[a, b] \subseteq \bigcup_{k=1}^{n} \left(\alpha_{i_k}, \beta_{i_k}\right) \cup \bigcup_{k=1}^{m} U_{\delta_{x_k}/2}(x_k)$$

Set $\delta = \frac{1}{2} \min_k \delta_{x_k}$ and let $(\Gamma, \eta), (\Lambda, \theta)$ be partitions of $[a, b]$ with $\|\Gamma\|, \|\Lambda\| < \delta$. We construct a refined partition

$$\Pi = \Gamma \cup \Lambda \cup \left([a, b] \cap \left(\left\{\alpha_{i_k}, \beta_{i_k}\right\}_{k=1}^{n} \cup \left\{ x_k \pm \frac{\delta_{x_k}}{2} \right\}_{k=1}^{m} \right) \right)$$

That is, Π contains all the points of Γ, Λ as well as all the endpoints of the intervals in our finite subcover which are in $[a, b]$. Write Π as

$$\Pi = (a = z_0 < z_1 < \cdots < z_\ell = b)$$

We define "pseudo-taggings" η', θ' of Π by simply inheriting the tags from η, θ; i.e, for any $0 \le r < \ell$, we set $\eta'_r = \eta_i, \theta'_r = \theta_j$ when $[z_r, z_{r+1}]$ is contained in the ith subinterval of Γ and the jth subinterval of Λ. With this notation, it is simple to check that

$$\sigma(f, \Gamma, \eta) = \sum_{k=0}^{\ell-1} f(\eta'_k)(z_{k+1} - z_k) \qquad \text{and} \qquad \sigma(f, \Lambda, \theta) = \sum_{k=0}^{\ell-1} f(\theta'_k)(z_{k+1} - z_k)$$

and thus

$$|\sigma(f, \Gamma, \eta) - \sigma(f, \Lambda, \theta)| \le \sum_{k=0}^{\ell-1} |f(\eta'_k) - f(\theta'_k)|(z_{k+1} - z_k)$$

Now, every subinterval $[z_k, z_{k+1}]$ must be contained either in some $(\alpha_i, \beta_i) \subseteq U$ or some $U_{\delta_{x_i}/2}(x_i)$, so we split up the above accordingly

$$\begin{aligned} &= \sum_{[z_k, z_{k+1}] \subseteq U} |f(\eta'_k) - f(\theta'_k)|(z_{k+1} - z_k) \\ &\quad + \sum_{[z_k, z_{k+1}] \subseteq U_{\delta_{x_i}/2}} |f(\eta'_k) - f(\theta'_k)|(z_{k+1} - z_k) \end{aligned}$$

In the first sum, the difference $|f(\eta'_k) - f(\theta'_k)| \le 2M$ and the total length of the subintervals $[z_k, z_{k+1}]$ is at most $|U| < \varepsilon/4M$, so we obtain

$$< \frac{\varepsilon}{2} + \sum_{[z_k, z_{k+1}] \subseteq U_{\delta_{x_i}/2}} |f(\eta'_k) - f(\theta'_k)|(z_{k+1} - z_k)$$

In the remaining sum, choose $m_k \in (z_k, z_{k+1})$; since η'_k, m_k belong to the same parent interval from Γ which has size $< \delta$, we have both $\eta'_k, m_k \in U_{\delta_{x_i}}$ for some i so $|f(\eta'_k) - f(m_k)| < \frac{\varepsilon}{4(b-a)}$. Likewise for θ'_k, m_k, so we obtain

$$|f(\eta'_k) - f(\theta'_k)| \le |f(\eta'_k) - f(m_k)| + |f(m_k) - f(\theta_k)| < \frac{\varepsilon}{2(b - a)}$$

and thus

$$|\sigma(f, \Gamma, \eta) - \sigma(f, \Lambda, \theta)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

By the Cauchy criterion, f is Riemann integrable over $[a, b]$. ■