The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of a test and/or for revealing the test materials to other students or to any unauthorized institution.

Please note, once this test has begun, you ***CANNOT*** *re-write it.*

Question 1 (5 points)

Say a function $f : [a, b] \to \mathbb{R}$ is *simple* if f has finite range and finitely many discontinuities. Prove that simple functions are exactly the linear combinations of characteristics of intervals contained in $[a, b]$ (i.e., functions of the form $\alpha_1\chi_{I_1} + \cdots + \alpha_n\chi_{I_n}$ for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $I_k \subseteq [a, b]$).

Hint. Try induction on the number of discontinuities.

Question 2 (5 points)

Prove that any simple function is Riemann integrable and find an expression for the integral. Use this to show that for any $f \in \mathfrak{R}[a, b]$, there exists a sequence of simple functions (f_n) such that $\lim_{n\to\infty} \int_a^b f_n(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x$.

Question 3 (5 points)

Let $f \in \mathfrak{R}[a,b]$ and $g : [c,d] \to \mathbb{R}$ be continuously differentiable such that $g'(x) \neq 0$ for all $x \in [c,d]$ and $g([c,d]) \subseteq [a,b]$. Prove that $f \circ g \in \mathfrak{R}[c,d]$.

Hint: Apply the Lebesgue criterion to reduce to showing $g^{-1}(D_f)$ is measure zero. Use continuity to argue g' has constant sign and conclude that g is monotone and thus injective. Finish by arguing that g^{-1} is Lipschitz and hence preserves measure zero sets.