MAT159 Test Solutions – Test #5

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Question 1

Say a function $f : [a, b] \to \mathbb{R}$ is *simple* if f has finite range and finitely many discontinuities. Prove that simple functions are exactly the linear combinations of characteristics of intervals.

Solution. There are two implications to prove.

($\Longrightarrow$) Let f be a simple function. We actually prove the result for functions over non-empty intervals I. We prove by induction on $n = |D_f|$ that f is a linear combination of characteristics of intervals.

The base case is $n = 0$. That is, f is continuous. We claim that f is constant. This is a general fact:

Lemma. *If $g : I \to \mathbb{R}$ is continuous over some non-empty interval I and takes on finitely many values, then g is constant.*

Proof. We prove the contrapositive. If g is not constant, there are $x, y \in [a, b], x \neq y$ such that $g(x) < g(y)$. The intermediate value theorem implies $[g(x), g(y)] \subseteq g(I)$ and thus g takes on infinitely many values. ■

So, if $|D_f| = 0$ then f is constantly α for some $\alpha \in \mathbb{R}$ and thus $f = \alpha\chi_I$.

Now, let $n \in \mathbb{N}$ be arbitrary and suppose any function with finite range and n discontinuities is a linear combination of characteristics of closed intervals.

Let $f : I \to \mathbb{R}$ be a function with finite range and $n + 1$ discontinuities. Let $a < b$ be the endpoints of I and $x_0 = \min D_f$ be the “first” point of discontinuity.

Now, $f|_{I\cap(-\infty,x_0)}$ is a simple function with 0 discontinuities and $f|_{I\cap(x_0,\infty)}$ is a simple function with n discontinuities so applying the induction hypothesis and adding $f(x_0)\chi_{[x_0,x_0]}$ completes the result.

($\Longleftarrow$) Let $f = \alpha_1\chi_{I_1} + \dots + \alpha_n\chi_n$ be a linear combination of characteristics of intervals I_k. Then f can take on at most 2^n values (one for each of the possible choices of the α_i to include in a sum) so f has finite range.

Moreover, it is easy to check that for any functions g, h, if $g + h$ is discontinuous at a point c then either h, g are discontinuous at c (consider the contrapositive to see why). This can be extended inductively to the sum of n functions. Thus, f can be discontinuous only at the points that any of the $\alpha\chi_{I_k}$ are discontinuous; since each $\alpha\chi_{I_k}$ is discontinuous at finitely many points so is f discontinuous at only finitely many points.

So, f is simple.

So, a function f is simple if and only if it is a linear combination of characteristics of closed intervals. ■

Question 2

Show that any simple function is Riemann integrable and finite an expression for the integral. Use this to show that for any $f \in \mathfrak{R}[a, b]$, there exists a sequence of simple functions (f_n) such that $\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$.

Solution.

Let $g : [a, b] \to \mathbb{R}$ be simple. By Question 1, $g = \alpha_1 \chi_{I_1} + \dots + \alpha_n \chi_{I_n}$ for $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $I_k = [a_k, b_k]$.

In Quiz 3, we saw that each χ_{I_k} is integrable over $[a, b]$ with $\int_a^b \chi_{I_k} = b_k - a_k$. By linearity of the Riemann integral, we have g integrable and

$$\begin{aligned}\int_a^b g(x)\,\mathrm{d}x &= \sum_{k=1}^{n} \alpha_k \int_a^b \chi_{I_k}(x)\,\mathrm{d}x \\ &= \sum_{k=1}^{n} \alpha_k (b_k - a_k)\end{aligned}$$

Now, let $f \in \mathfrak{R}[a, b]$ and $S = \int_a^b f(x)\,\mathrm{d}x$. Fix $n \in \mathbb{N}$. Choose a partition $\Gamma_n = x_0 < \dots < x_m$ of $[a, b]$ such that $\left|\overline{S}(f, \Gamma_n) - S\right| < 1/n$. Define

$$f_n = \sum_{i=0}^{m-1} M_{x_i, x_{i+1}} \chi_{[x_i, x_{i+1}]}$$

Since f_n is a linear combination of characteristics of intervals, Question 1 gives us f_n simple. Moreover, we have f_n integrable with

$$\int_a^b f_n(x)\,\mathrm{d}x = \sum_{i=0}^{m-1} M_{x_i, x_{i+1}} (x_{i+1} - x_i) = \overline{S}(f, \Gamma_n)$$

So, fix $\varepsilon > 0$ and choose N sufficiently large so that $1/N < \varepsilon$. If $n > N$, we have

$$\left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| = \left|\overline{S}(f, \Gamma_n) - S\right| < \frac{1}{n} < \frac{1}{N} < \varepsilon$$

as needed. ■

Question 3

Let $f \in \mathfrak{R}[a,b]$ and $g : [c,d] \to \mathbb{R}$ be continuously differentiable such that $g'(x) \neq 0$ for all $x \in [c,d]$ and $g([c,d]) \subseteq [a,b]$. Show that $f \circ g \in \mathfrak{R}[c,d]$.

Proof. Note that

$$\begin{aligned} f \circ g \text{ discontinuous at } c &\implies f \text{ discontinuous at } g(c) \\ &\implies c \in g^{-1}(D_f) \end{aligned}$$

where the first implication is because if f is continuous at $g(c)$ then, as g is continuous, $f \circ g$ must be continuous at c.

So, $D_f(f \circ g) \subseteq g^{-1}(D_f)$. Thus, if we show $g^{-1}(D_f)$ has measure zero, we can conclude by the Lebesgue criterion that $f \circ g$ is integrable.

First, observe that as g' is never zero but continuous, g' must in fact have constant sign. So, g is monotone and thus injective. So, $g^{-1} : g([c,d]) \to [c,d]$ exists, and the problem now reduces to showing the image of D_f under g^{-1} is null.

To this end, we show g^{-1} is Lipschitz. By continuity of g', there exists a minimum value $m \neq 0$ for $|g'|$. So, fix $u, v \in g([c,d])$ and write $u = g(x), v = g(y)$. By the mean value theorem, $|g(x) - g(y)| = |g'(\zeta)(x-y)|$ so

$$\begin{aligned} |g^{-1}(u) - g^{-1}(v)| &= |x - y| \\ &= \left(\frac{1}{|g'(\zeta)|}\right)|g(x) - g(y)| \\ &\leq \frac{1}{m}|u - v| \end{aligned}$$

and thus g^{-1} is Lipschitz with constant $L := 1/m$.

Finally, we aim to show $g^{-1}(D_f)$ has measure zero. Let $\varepsilon > 0$ be given and choose an open cover U of D_f with total length at most ε/L. As g^{-1} is Lipschitz, we have $|g^{-1}(I)| \leq L|I|$ for any I. Moreover, if I is open so is $g^{-1}(I)$ as g is continuous.

So, we have $g^{-1}(U) := \{g^{-1}(I) : I \in U\}$ an open cover of $g^{-1}(D_f)$ with

$$|g^{-1}(U)| = \sum_{I \in U} |g^{-1}(U)| \leq L \sum_{I \in U} |I| < \varepsilon$$

Thus, $g^{-1}(D_f)$ has measure zero so $f \circ g$ is integrable on $[c,d]$. ■

Remark. A key step in this proof is reframing the $g^{-1}(D_f)$ from “the g-preimage of D_f” to “the g^{-1}-image of D_f”. The same exact notation $g^{-1}(D_f)$ is used to reference two different ideas which happen to coincide in the context of this argument. Make sure that you can point to each use of $g^{-1}(D_f)$ and identify whether, in that context, it is better thought of as image of preimage.

Bonus problem. Are there different sets of assumptions on g that can make the result true? For example, is the claim true if g is merely differentiable but not continuously? What if g is only continuous?