MAT159 Test Solutions – Test #6

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Question 1

Find all continuous functions $f : [-1, 1] \to \mathbb{R}$ satisfying

$$\int_a^{x^2} f(t)\,\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t$$

for all $x \in [-1, 1]$, where $a \in [-1, 1]$ is a fixed constant.

Solution. It is easy to see that $f = 0$ works. We claim that this is the only solution.

Suppose $f : [-1, 1] \to \mathbb{R}$ is continuous and satisfies the equation. Setting

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

our equation becomes $F(x^2) = F(x)$ for all $x \in [-1, 1]$. We claim that F is constant; since F is continuous by the FTC, it suffices to show $F(x) = F(0)$ for all $-1 < x < 1$.

So, fix x with $|x| < 1$. We have $F(x) = F(x^2) = F(x^4) = \ldots$ so we can write

$$F(x) = \lim_{n \to \infty} F\left(x^{2^n}\right)$$

which, by continuity of F, becomes

$$F(x) = F\left(\lim_{n \to \infty} x^{2^n}\right) = F(0)$$

So, F is constant. By the FTC, $f = F'$, so f is identically zero. ∎

Bonus Problem. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_0^{x + \int_0^x f(s)\,\mathrm{d}s} f(t)\,\mathrm{d}t = \int_0^x f(t)\,\mathrm{d}t$$

Question 2

Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable with $f(a) = 0$. Show that

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \frac{(b-a)^2}{2} \sup_{x \in [a,b]} |f'(x)|$$

Solution. Let $M = \sup_{x \in [a,b]} |f'(x)|$. Note that continuity of f' guarantees M finite.

Now, fix $x \in [a,b]$; we have by the FTC and the assumption $f(a) = 0$ that

$$f(x) = f(x) - f(a) = \int_a^b f'(t)\,\mathrm{d}t$$

The MVT gives the last integral equal to $f'(c)(x-a)$ for some $c \in (a,x)$ so we have

$$|f(x)| \leq M(x-a)$$

for all $x \in [a,b]$. Thus,

$$\begin{aligned}
\left| \int_a^b f(x)\,\mathrm{d}x \right| &\leq \int_a^b |f(x)|\,\mathrm{d}x \\
&\leq \int_a^b M(x-a)\,\mathrm{d}x \\
&= M\left(\frac{x^2}{2} - ax\right)\Bigg|_a^b \\
&= M\left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2\right) \\
&= \frac{(b-a)^2}{2} M
\end{aligned}$$

as needed. ■

Question 3

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous satisfying for all $x \in \mathbb{R}$ there exists $\delta > 0$ such that for all $0 < h < \delta$, we have

$$\int_x^{x+h} f(t)\,\mathrm{d}t = 0$$

Show that f is identically zero.

Solution.

Fix $x \in \mathbb{R}$. Let $\delta > 0$ be as given in the problem statement. For each $n = 2, \ldots$, choose some $c_n \in (x, x + \delta/n)$ such that

$$\int_x^{x+\delta/n} f(t)\,\mathrm{d}t = f(c_n)\delta_n$$

The existence of such a c_n is by the integral mean value theorem. It is clear that each $f(c_n) = 0$ and that $c_n \longrightarrow x$, so by continuity of f, we have

$$f(x) = f\left(\lim_{n\to\infty} c_n\right) = \lim_{n\to\infty} f(c_n) = 0$$

so f is identically zero. ∎