MAT159 Test Solutions – Test #7

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2026-03-12

Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be T-periodic and integrable. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \int_x^{x+T} f(s)\,\mathrm{d}s$. What is $g'(x)$?

Solution.

We claim $g'(x) = 0$ for which it suffices to show g is constant. Indeed, we'll show $g(x) = \int_0^T f(s)\,\mathrm{d}s$ for all $x \in \mathbb{R}$.

So, fix $x \in \mathbb{R}$. Let $k = \lfloor x/T \rfloor$ and make the substitution $s \to s - kT$ to get

$$g(x) = \int_x^{x+T} f(s)\,\mathrm{d}s = \int_{x-kT}^{x-kT+T} f(s+kT)\,\mathrm{d}s = \int_{x-kT}^{x-kT+T} f(s)\,\mathrm{d}s$$

where the last equality is due to periodicity of f.

Since $k = \lfloor x/T \rfloor$, we have $k \leq x/T$ and thus $x - kT + T \geq T$ so we split our integral

$$g(x) = \int_{x-kT}^{x-kT+T} f(s)\,\mathrm{d}s = \int_{x-kT}^{T} f(s)\,\mathrm{d}s + \int_{T}^{x-kT+T} f(s)\,\mathrm{d}s$$

In the second integral, we substitute $s \to s + T$ to get

$$\int_{T}^{x-kT+T} f(s)\,\mathrm{d}s = \int_{0}^{x-kT} f(s+T)\,\mathrm{d}s = \int_{0}^{x-kT} f(s)\,\mathrm{d}s$$

so we have

$$g(x) = \int_{x-kT}^{T} f(s)\,\mathrm{d}s + \int_{T}^{x-kT+T} f(s)\,\mathrm{d}s = \int_{x-kT}^{T} f(s)\,\mathrm{d}s + \int_{0}^{x-kT} f(s)\,\mathrm{d}s = \int_{0}^{T} f(s)\,\mathrm{d}s$$

as claimed. Since g is constant, $g'(x) = 0$. ■

Question 2

Let $f \in \mathfrak{R}[a,b]$ and F be a primitive of f. Show that $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

Solution.

We show this by definition of the Riemann integral. Fix $\varepsilon > 0$. Since f is integrable, there is $\delta > 0$ such that for any $(\Gamma, \eta), (\Lambda, \theta) \in \Omega^*[a,b]$ with $\|\Gamma\|, \|\Lambda\| < \delta$, we have

$$|\sigma(f,\Gamma,\eta) - \sigma(f,\Lambda,\theta)| < \varepsilon$$

Let $(\Gamma,\eta) \in \Omega^*[a,b]$ with $\|\Gamma\| < \delta$. Write $\Gamma = x_0 < x_1 < \dots < x_n$.

On each subinterval $[x_i, x_{i+1}]$, we have F differentiable so by the mean value theorem, there is $\theta_i \in (x_i, x_{i+1})$ such that

$$F(x_{i+1}) - F(x_i) = F'(\theta_i)(x_{i+1} - x_i) = f(\theta_i)(x_{i+1} - x_i)$$

Let θ denote the sequence of θ_i. Now, by our choice of δ, we have

$$|\sigma(f,\Gamma,\eta) - \sigma(f,\Gamma,\theta)| < \varepsilon$$

but we have

$$\sigma(f,\Gamma,\theta) = \sum_{i=0}^{n-1} f(\theta_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i) = F(x_n) - F(x_0) = F(b) - F(a)$$

so that

$$|\sigma(f,\Gamma,\eta) - (F(b) - F(a))| < \varepsilon$$

which shows $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, as needed. ■

Question 3

Show that if $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then f has bounded variation.

Solution.

Taking $\varepsilon = 1$, we have by absolute continuity some $\delta > 0$ such that if $\{(a_i, b_i)\}_{i=1}^{k}$ are a finite collection of intervals with total length less than δ, then $\sum_{i=1}^{k} |f(b_i) - f(a_i)| < 1$.

Choose N sufficiently large so that $(b - a)/N < \delta$.

Fix any partition $\Gamma = (x_0 < \cdots < x_n) \in \Omega[a, b]$ and consider $\Gamma' = \Gamma \cup \left\{a + i\frac{b-a}{N} : 0 \leq i \leq N\right\}$.

Since Γ' refines Γ, the variation of f over Γ is at most the variation of f over Γ' (this is an application of the triangle inequality).

Write $\Gamma' = y_0 < \cdots < y_m$ and let $i_0, ..., i_N \in \{0, ..., m\}$ be the indices of the $a + i(b - a)/N$ points. Then,

$$\begin{aligned}\sum_{i=0}^{m-1} |f(y_{i+1}) - f(y_i)| &= \sum_{k=0}^{N-1} \left(\sum_{i_k \leq i < i_{k+1}} |f(y_{i+1}) - f(y_i)| \right) \\ &< \sum_{k=0}^{N-1} 1 = N\end{aligned}$$

so the variation over any partition is bounded and thus f has bounded variation. ■