The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of a test and/or for revealing the test materials to other students or to any unauthorized institution.

Please note, once this test has begun, you ***CANNOT*** *re-write it.*

Question 1 (5 points)

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Show that if the series $\sum_{i=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. Decide if the converse is true or false and prove your answer.

Question 2 (5 points)

Let $N \in \mathbb{N}$ and $f : [N, \infty) \to \mathbb{R}$ be monotone decreasing. Prove that

$$\sum_{n=N}^{\infty} f(n) < \infty \iff \int_N^{\infty} f(t)\ \mathrm{d}t < \infty$$

where $\int_N^{\infty} f(t)\ \mathrm{d}t$ is defined as $\lim_{x\to\infty} \int_N^{x} f(t)\ \mathrm{d}t$.

Hint: Apply the comparison test twice, comparing $f(n)$ with the integrals of f over $[n-1, n]$ and $[n, n+1]$ respectively.

Question 3 (5 points)

For $k = 1, 2, \ldots$, let $\log_k(x)$ denote the k-fold composition of log. That is, $\log_1(x) = \log(x)$ and $\log_{k+1}(x) = \log_k(\log(x))$. Let N_k denote the least integer such that $\log_k(N_k)$ is well-defined and positive.

Show that for all $k \geq 1$, the series $\displaystyle\sum_{n=N_k}^{\infty} \frac{1}{n \prod_{i=1}^{k} \log_i(n)}$ is divergent.