MAT159 Test Solutions – Test #8

Mustafa Motiwala

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Question 1

Let (a_n) be a sequence of real numbers. Show that if $\sum_{i=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty}|a_{n+1} - a_n| = 0$. Decide the converse.

Proof.

For *any* convergent sequence (a_n), we have $\lim_{n\to\infty}|a_{n+1} - a_n| = 0$ because convergent implies Cauchy.

It was shown in class that when $\sum_{i=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$ (this is by applying the previous claim to the sequence of partial sums).

For the converse, consider $a_n = 1/n$. Then $\lim_{n\to\infty}|a_{n+1} - a_n| = 0$ but the series $\sum_{i=1}^{\infty} 1/n$ diverges. ■

Question 2

Let $N \in \mathbb{N}$ and $f : [N, \infty) \to \mathbb{R}$ be monotone decreasing. Show that

$$\sum_{n=N}^{\infty} f(n) < \infty \Longleftrightarrow \int_{N}^{\infty} f(t)\,\mathrm{d}t < \infty$$

Proof. Observe that $\displaystyle\int_{N}^{\infty} f(t)\,\mathrm{d}t = \sum_{n=N}^{\infty} \int_{n}^{n+1} f(t)\,\mathrm{d}t = \sum_{n=N+1}^{\infty} \int_{n-1}^{n} f(t)\,\mathrm{d}t$ (by this we mean either everything diverges or everything converges to the same value).

Thus, it will suffice to compare the terms of

$$\sum_{n=N}^{\infty} \int_{n}^{n+1} f(t)\,\mathrm{d}t \le \sum_{n=N}^{\infty} f(n) \le f(N) + \sum_{n=N+1}^{\infty} \int_{n-1}^{n} f(t)\,\mathrm{d}t$$

This follows immediately from monotonicity as for all $n \ge N$,

$$f(x) \le f(n) \text{ for } x > n \Longrightarrow \int_{n}^{n+1} f(t)\,\mathrm{d}t \le f(n)$$
$$f(n) \le f(x) \text{ for } n > x \Longrightarrow f(n) \le \int_{n-1}^{n} f(t)\,\mathrm{d}t$$

The comparison test gives the desired equivalence. ■

Remark. We don't assume integrability of f because it follows from monotonicity. Recall that monotone functions have countably many discontinuities (we showed this in the last test of MAT157) and so are integrable everywhere by Lebesgue's criterion.

Remark. You may see this result stated with the additional assumption that $f(x) \ge 0$ for all x. The assumption is unnecessary since if $f(x) < 0$ for some x, then the monotone condition implies that both the series and integral must diverge (why?).

Question 3

For $k \geq 1$, let $\log_k(x)$ denote the k-fold composition of log and N_k denote the least integer such that $\log_k(N_k)$ is well-defined and positive. Show that for all $k \geq 1$, the series $\displaystyle\sum_{n=N_k}^{\infty} \frac{1}{n \prod_{i=1}^{k} \log_i(n)}$ is divergent.

Proof. Fix $k \geq 1$. We will apply the result of Question 2. To this end, let $f : [N_k, \infty)$ denote the function

$$f(x) = \frac{1}{x \prod_{i=1}^{k} \log_i(x)}$$

To apply Question 2, we need to know f is monotone decreasing. This should be clear, as log is increasing, compositions of increasing functions are increasing, and products of positive increasing functions are increasing. Thus, the denominator in f is increasing, so f is decreasing.

We want to show the series $\sum_{n=N_k}^{\infty} f(n)$ is divergent. By Question 2, it suffices to show the improper integral $\int_{N_k}^{\infty} f(t)\,\mathrm{d}t$ is divergent. To evaluate this integral, observe that f is actually the derivative of $\log_{k+1}$. This can be seen by induction where the inductive step is simply the computation:

$$\frac{\mathrm{d}}{\mathrm{d}x} \log_{k+1}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log(\log_k(x)) = \frac{1}{\log_k(x)} \left(\frac{\mathrm{d}}{\mathrm{d}x} \log_k(x) \right)$$

So,

$$\begin{aligned} \int_{N_k}^{\infty} f(t)\,\mathrm{d}t &= \lim_{x \to \infty} \int_{N_k}^{x} \left(\frac{\mathrm{d}}{\mathrm{d}x} \log_{k+1}(t)\,\mathrm{d}t \right) \\ &= \lim_{x \to \infty} \left(\log_{k+1}(x) - \log_{k+1}(N_k) \right) \\ &= \infty \end{aligned}$$

Since the integral diverges, so does the series $\sum_{n=N_k}^{\infty} f(n)$. Where the last equality is because $\log_{k+1}$ is unbounded (the composition of unbounded increasing functions is unbounded). ■

Remark. In class, you showed the p-test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$. This result shows that there are infinitely many functions f that are "in-between" $\frac{1}{n}$ and $\frac{1}{n^{1+\varepsilon}}$ for all $\varepsilon > 0$ such that $\sum_{n=1}^{\infty} f(n)$ diverges.

Bonus. Show that for all $k \geq 1$ and all $\varepsilon > 0$, the series $\displaystyle\sum_{n=N_k}^{\infty} \frac{1}{n(\log_k(n))^{1+\varepsilon} \prod_{i=1}^{k-1} \log_i(n)}$ converges.