# MAT159 Test Solutions – Test #9

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## Question 1

Find all $c > 0$ for which

$$\sum_{n=1}^{\infty} \frac{n!}{(1+c)(2+c)\cdots(n+c)}$$

converges.

*Solution.* Let $a_n = (n!)/\prod_{k=1}^{n}(k+c)$. We apply the Kummer test with modulus $c_n = n$, also known as Raabe's test. We have

$$\begin{aligned}
K_n &= n\left(\frac{a_n}{a_{n+1}}\right) - (n+1) \\
&= n\left(\frac{n!}{\prod_{k=1}^{n}(k+c)}\right)\left(\frac{\prod_{k=1}^{n+1}(k+c)}{(n+1)!}\right) - (n+1) \\
&= \frac{n(n+1+c)}{n+1} - (n+1) \\
&= \frac{(n+1)n + nc - (n+1)^2}{n+1} \\
&= \frac{(n+1)(n-(n+1)) + nc}{n+1} \\
&= c\frac{n}{n+1} - 1
\end{aligned}$$

so $\lim_{n\to\infty} K_n = c - 1$. By the Kummer test, we have convergence when $c - 1 > 0 \Leftrightarrow c > 1$ and divergence when $c - 1 < 0 \Leftrightarrow c < 1$.

When $c = 1$, we have $a_n = \frac{1}{n+1}$ and the series diverges.

Thus, the series in question converges if and only if $c > 1$. ■

## Question 2

Decide whether $\sum_{n \text{ palindrome}} \frac{1}{n}$ converges.

*Solution.* We will show that the series converges.

First, let us count the number of $k$-digit palindromes, call it $N_k$. Such a number has $k$ digits $d_1 \cdots d_k$ and can be anything, subject to the constraints $d_1 \neq 0$ (the leading digit may not be zero) and $d_1 = d_k, d_2 = d_{k-1}, d_3 = d_{k-2}, \ldots$ (to ensure the number is the same backwards and forwards). Thus, there are 9 possible values for $d_1$ and 10 possible values for the remaining digit pairs, giving $N_k = 9 \cdot 10^{\lceil \frac{k}{2} \rceil - 1}$.

Furthermore, note that any $k$-digit number is at most $10^k$. Thus, grouping terms by the number of digits, we get

$$\begin{aligned} \sum_{n \text{ palindrome}} \frac{1}{n} &= \sum_{k=1}^{\infty} \sum_{n \text{ a } k\text{-digit palindrome}} \frac{1}{n} \\ &\leq \sum_{k=1}^{\infty} \frac{N_k}{10^k} && \text{there are } N_k \ k\text{-digit palindromes, each} \leq \tfrac{1}{10^k} \\ &\leq \sum_{k=1}^{\infty} \frac{9 \cdot 10^{\frac{k}{2}}}{10^k} \\ &= \sum_{k=1}^{\infty} 9 \left( \frac{1}{\sqrt{10}} \right)^k \end{aligned}$$

This final series converges as it is a geometric series with ratio less than 1. By the comparison test, the series of reciprocal palindromes converges. ■

**Bonus.** What does it converge to?

## Question 3

Let $A = \sum_{n=0}^{\infty} a_n$ be absolutely convergent. Let $B = \sum_{n=0}^{\infty} b_n$ be convergent. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Show that $\sum_{n=0}^{\infty} c_n = AB$.

*Solution.* Let $(A_n), (B_n), (C_n)$ be partial sums of the $(a_n), (b_n), (c_n)$ series. First, we compute

$$\begin{aligned} C_n &= \sum_{i=0}^{n} \left( \sum_{k=0}^{i} a_k b_{i-k} \right) \\ &= \sum_{i=0}^{n} a_{n-i} \left( \sum_{k=0}^{i} b_k \right) = \sum_{i=0}^{n} a_{n-i} B_i \\ &= A_n B + \sum_{i=0}^{n} a_{n-i}(B_i - B) && \text{adding } A_n B - A_n B \end{aligned}$$

from which we obtain

$$C_n - AB = (A_n - A)B + \sum_{i=0}^{n} a_{n-i}(B_i - B) \tag{1}$$

Now, fix $\varepsilon > 0$. First choose $N_A \in \mathbb{N}$ such that

$$n > N_A \implies |A_n - A| < \frac{\varepsilon/3}{1 + |B|} \tag{2}$$

Also, choose $N_B \in \mathbb{N}$ such that

$$n > N_B \implies |B_n - B| < \frac{\varepsilon/3}{1 + \sum_{k=0}^{\infty} |a_k|} \tag{3}$$

Note that we use absolute convergence of the $(a_n)$ sequence here. Finally, since the series $\sum_{n=0}^{\infty} a_n$ converges, the individual terms $(a_n)$ converge to 0, so we can choose $N_a \in \mathbb{N}$ such that

$$n > N_a \implies |a_n| < \frac{\varepsilon/3}{1 + (N_B + 1) \max_{i=0}^{N_B} |B_i - B|} \tag{4}$$

Take $N = \max\{N_A, N_B + N_a\}$. Let $n > N$. Then,

$$\begin{aligned} |C_n - AB| &= \left| (A_n - A)B + \sum_{i=0}^{n} a_{n-i}(B_i - B) \right| && \text{Equation (1)} \\ &\leq |A_n - A||B| + \sum_{i=0}^{N_B} |a_{n-i}||B_i - B| + \sum_{i=N_B+1}^{n} |a_{n-i}||B_i - B| \\ &< \frac{\varepsilon}{3} + \left( \max_{i=0}^{N_B} |B_i - B| \right) \sum_{i=0}^{N_B} |a_{n-i}| + \sum_{i=N_B+1}^{n} |a_{n-i}||B_i - B| && \text{Equation (2)} \end{aligned}$$

In the middle term, since $i \leq N_B, n - i > (N_B + N_a) - i \geq N_a$ so Equation (4) gives

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=N_B+1}^{n} |a_{n-i}||B_i - B|$$

Finally, we have each $|B_i - B|$ bounded by Equation (3) and the contribution of the $|a_{n-i}|$ is bounded by the sum of the entire series so we get

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus, $C_n \longrightarrow AB$, as needed. ■

**Remark.** It might not immediately seem like it, but the roles of $A$ and $B$ are symmetric. We need one of the series $\sum_n a_n, \sum_n b_n$ to converge absolutely but it does not matter which one.