Metrizing Pointwise Convergence

L. Gauvreau, M. Motiwala

Abstract

We show how to metrize pointwise convergence on the unit ball of the dual of a separable normed vector space. Under our metric, the unit ball of the dual becomes compact.

§1 Introduction

Recall the operator norm on bounded linear maps, which computes the maximal norm attained by a linear map on the unit ball of its domain. It is well known that the operator norm induces the topology of *uniform convergence*. Another natural notion of convergence of functions is *pointwise convergence*, wherein a *sequence* of functions $f_n \to f$ if and only if $f_n(x) \to f(x)$ for all x. One may well ask whether the topology of pointwise convergence is metrizable, that is, does there exist a metric d on a set of functions satisfying the property that $f_n \to f$ pointwise if and only if f_n converges to f under d.

Under certain conditions, we are able to answer this question in the affirmative. Specifically, we show if X is a separable normed vector space, then pointwise convergence on the closed unit ball of X^* under the operator norm is metrizable, and moreover that it is compact under the induced topology. Here, X^* denotes the continuous dual space of X, that is, the space of bounded, linear functions from $X \to \mathbb{R}$.

We also discuss necessary and sufficient conditions for metrizing pointwise convergence on all of X^*.

§2 Preliminary Results

We establish a few technical lemmas that will be helpful for the main result.

Lemma 2.1. *For (X, d) a metric space, the function*

$$\hat{d} : X \times X \to \mathbb{R} \qquad \hat{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is a topologically equivalent metric to d.

Proof. Let us first verify that $\hat{d}$ is indeed a metric. For any $x, y \in X$, notice that

$$\hat{d}(x,y) = 0 \iff \frac{d(x,y)}{d(x,y)+1} = 0 \iff d(x,y) = 0 \iff x = y$$

Furthermore, since $d(x,y) \geq 0$, it follows that $d(x,y)+1 \geq 1$, so clearly $\hat{d}(x,y) \geq 0$. Symmetry follows immediately from the symmetry of d. Note that for any $x, y \in X$, we have that $0 \leq d(x,y) < d(x,y)+1$, so

$$0 \leq \frac{d(x,y)}{d(x,y)+1} < 1$$

which shows that $\hat{d}$ is bounded by 1. To assist in proving the triangle inequality, we start by defining the function $f : [0, \mathbb{R}) \to [0, \mathbb{R})$ by $f(t) = \frac{t}{t+1}$. Notice that f is differentiable with derivative

$$f'(t) = \frac{1(t+1) - t(1)}{(t+1)^2} = \frac{1}{(t+1)^2} > 0$$

so $f'(t) > 0$ for all $t \in [0, \infty)$. We know this implies that f is an increasing function. Now fix $x, y, z \in X$. Since f is increasing, we have that

$$d(x,z) \leq d(x,y) + d(y,z) \implies f(d(x,z)) \leq f(d(x,y) + d(y,z))$$

Using the triangle inequality d satisfies, it follows that

$$\begin{aligned} \frac{d(x,z)}{d(x,z)+1} &\leq \frac{d(x,y)+d(y,z)}{d(x,y)+d(y,z)+1} \\ &= \frac{d(x,y)}{d(x,y)+d(y,z)+1} + \frac{d(y,z)}{d(x,y)+d(y,z)+1} \\ &\leq \frac{d(x,y)}{d(x,y)+1} + \frac{d(y,z)}{d(y,z)+1} = \hat{d}(x,y) + \hat{d}(y,z) \end{aligned}$$

Note that the last inequality holds as we are dividing both terms by a smaller number. This shows that $\hat{d}$ is indeed a metric. We now show that d and $\hat{d}$ are topologically equivalent. Given $x \in X$ and $r > 0$, we'll let $\hat{B}(x,r) = \{y \in X : \hat{d}(x,y) < r\}$. Fix $x \in X$, $r > 0$, and consider $\hat{B}(x,r)$. Since $\hat{d}$ is bounded by 1, we may assume that $r < 1$, as otherwise the open ball would equal all of X. Notice that

$$\begin{gathered} \hat{d}(x,y) < r \iff \frac{d(x,y)}{d(x,y)+1} < r \iff d(x,y) < rd(x,y) + r \\ \iff (1-r)d(x,y) < r \iff d(x,y) < \frac{r}{1-r} \end{gathered}$$

which shows that $\hat{B}(x,r) = B(x, \frac{r}{1-r})$, so $\hat{B}(x,r)$ is open in (X,d). Similarly, for any $x \in X$ and $r > 0$, we have that

$$\hat{d}(x,y) < \frac{r}{r+1} \iff \frac{d(x,y)}{d(x,y)+1} < \frac{r}{r+1} \iff rd(x,y) + d(x,y) < rd(x,y) + r \iff d(x,y) < r$$

showing $B(x,r) = \hat{B}(x, \frac{r}{r+1})$, so $B(x,r)$ is open in $(X, \hat{d})$. Since $(X, \hat{d})$ and (X,d) have the same open balls, they must have the same open sets, so d and $\hat{d}$ are topologically equivalent.

□

The following lemma shows how to metrize pointwise convergence of *sequences* and will do a lot of the heavy lifting for our main result.

Lemma 2.2. *Let (X_n, d_n) be a sequence of metric spaces. Take $X = \prod_{i \in \mathbb{N}} X_i$ and define $\rho : X \times X \to \mathbb{R}$ by*

$$\rho((x_k),(y_k)) = \sum_{k=1}^{\infty} \frac{1}{2^k} \hat{d}_k(x_k, y_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(x_k,y_k)}{1 + d_k(x_k,y_k)}$$

Then, ρ is a well-defined metric on X. Moreover, the topology induced by ρ is that of pointwise convergence of sequences. That is, a sequence $(\overline{x_n})$ in X converges to $\overline{y}$ with respect to ρ if and only if the sequences $\left(\overline{x_{n,k}}\right)_{n=1}^{\infty} \to \overline{y}_k$ for all $k \in \mathbb{N}$.

Proof. Well-definedness of ρ is a quick consequence of the Basic Comparison Test: for each $k \in \mathbb{N}$, we have

$$0 \leq \frac{1}{2^k}\hat{d}_k(x_k, y_k) \leq \frac{1}{2^k}$$

so – since $\sum_{k=1}^{\infty} 2^{-k}$ converges – so does $\rho((x_k), (y_k))$. Thus, ρ is well-defined.

Now, we show that ρ is a metric. Positive-definiteness and symmetry are immediate consequences of positive-definiteness and symmetry of the metrics on X_n. To see that the triangle inequality holds, note that for $(x_k), (y_k), (z_k) \in X$, we have

$$\begin{aligned}
\rho((x_k), (z_k)) &= \sum_{k=1}^{\infty} \frac{1}{2^k}\hat{d}_k(x_k, z_k) \\
&\leq \sum_{k=1}^{\infty} \frac{1}{2^k}\Big(\hat{d}_k(x_k, y_k) + \hat{d}_k(y_k, z_k)\Big) && \text{by } \textbf{Lemma } 2.1 \\
&= \sum_{k=1}^{\infty} \frac{1}{2^k}\hat{d}_k(x_k, y_k) + \sum_{k=1}^{\infty} \frac{1}{2^k}\hat{d}_k(y_k, z_k) \\
&= \rho((x_k), (y_k)) + \rho((y_k), (z_k))
\end{aligned}$$

Thus, ρ is a metric on X.

Now, let $(\overline{x_n})$ be a sequence in X and $\overline{y} \in X$. We show that convergence of $(\overline{x_n})$ under ρ is equivalent to pointwise convergence of the component sequences.

$\Longrightarrow$ Suppose $(\overline{x_n})$ converge to $\overline{y}$ under ρ.

Fix $k \in \mathbb{N}$. By **Lemma** 2.1, it suffices to show convergence of the component sequence $\big(\overline{x_{n,k}}\big)$ under the bounded metric $\hat{d}_k$ to show convergence in d_k.

So, let $\varepsilon > 0$ be given, and choose $N \in \mathbb{N}$ so that if $n > N$, then $\rho(\overline{x_n}, \overline{y}) < \varepsilon/2^k$.

Let $n > N$. Then, since $\rho(\overline{x_n}, \overline{y})$ is a sum of non-negative terms bounded by $\varepsilon/2^k$, each term in the sum has the same bound. In particular, we have

$$\frac{1}{2^k}\hat{d}_k\big(\overline{x_{n,k}}, \overline{y}_k\big) < \varepsilon/2^k$$

which gives $\hat{d}_k\big(\overline{x_{n,k}}, \overline{y}_k\big) < \varepsilon$.

Thus, convergence under ρ implies pointwise convergence of each component sequence.

$\Longleftarrow$ Now, suppose each component sequence $\big(\overline{x_{n,k}}\big)_{n=1}^{\infty}$ converges to $\overline{y}_k$ under d_k. By **Lemma** 2.1, we can assume convergence of the component sequences under $\hat{d}_k$.

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that

$$\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2}$$

and set $C = \sum_{k=1}^{N} 2^{-k}$.

By convergence of the component sequences, we can choose for each $1 \leq i \leq N$, some $M_i \in \mathbb{N}$ so that if $n > M_i$, then

$$\hat{d}_k\big(\overline{x_{n,k}}, \overline{y}_k\big) < \frac{\varepsilon}{2C}$$

Finally, we take $K = \max_{1 \le i \le N} M_i$. So, if $n > K$, then

$$\begin{aligned}
\rho(\overline{x_n}, \overline{y}) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \hat{d}_k\left(\overline{x_{n}}_{,k}, \overline{y}_k\right) \\
&= \sum_{k=1}^{N} \frac{1}{2^k} \hat{d}_k\left(\overline{x_{n}}_{,k}, \overline{y}_k\right) + \sum_{k=N+1}^{\infty} \frac{1}{2^k} \hat{d}_k\left(\overline{x_{n}}_{,k}, \overline{y}_k\right) \\
&\le \sum_{k=1}^{N} \frac{1}{2^k} \hat{d}_k\left(\overline{x_{n}}_{,k}, \overline{y}_k\right) + \sum_{k=N+1}^{\infty} \frac{1}{2^k} \\
&< \sum_{k=1}^{N} \frac{1}{2^k} \hat{d}_k\left(\overline{x_{n}}_{,k}, \overline{y}_k\right) + \frac{\varepsilon}{2} \\
&< \sum_{k=1}^{N} \frac{1}{2^k} \frac{\varepsilon}{2C} + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{2C} \sum_{k=1}^{N} \frac{1}{2^k} + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{2C} C + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus, pointwise convergence of $(\overline{x_n})$ to $\overline{y}$ implies convergence under ρ.

So, we can metrize pointwise convergence on sequence spaces. $\square$

Lemma 2.3. *Let (X_n, d_n) be a sequence of compact metric spaces. Take $X = \prod_{i \in \mathbb{N}} X_i$ and equip X with the metric ρ constructed in **Lemma** 2.2. Then, (X, ρ) is compact.*

Proof. We show sequential compactness of X.

Let $(\overline{x_k})$ be a sequence in X. We recursively define a sequence $\left(\widetilde{t_n}\right)$ of subsequences of $(\overline{x_k})$ as follows:

- By compactness of X_1, the sequence $(x_{k,1})_{k=1}^{\infty}$ has some convergent subsequence $\left(x_{k_j,1}\right)$ converging to some $p_1 \in X_1$. We take $\widetilde{t_1} = \left(\overline{x_{k_j}}\right)$. That is, $\widetilde{t_1}$ is a subsequence of $(\overline{x_k})$ such that the sequence obtained by taking the first coordinate of each sequence in $\widetilde{t_1}$ converges to p_1.
- Now, given a subsequence $\widetilde{t_n}$ of $(\overline{x_k})$, the sequence $\left(\widetilde{t_n}_{k,n+1}\right)_{k=1}^{\infty}$ obtained by taking the $(n+1)$th coordinate of each sequence in $\widetilde{t_n}$ is a sequence in X_{n+1}. By compactness, this sequence has a subequence converging to some $p_{n+1} \in X_{n+1}$. So, we take $\widetilde{t_{n+1}}$ to be the subsequence of $\widetilde{t_n}$ corresponding to this subsequence. In other words, the sequence obtained by taking the $(n+1)$th coordinate of each sequence in $\widetilde{t_{n+1}}$ will converge to p_{n+1}.

It is easy to see by induction the following:

1. For each $n \in \mathbb{N}$, the sequence of the first n coordinates of each sequence in $\widetilde{t_n}$ converges to some $(p_1, \ldots, p_n) \in \prod_{k=1}^{n} X_k$.
2. These limits are the *same* for all $\widetilde{t_n}$. In other words, for $n > m$, the sequence of the first m coordinates of each sequence in $\widetilde{t_n}$ converges to the *same* $(p_1, \ldots, p_m) \in \prod_{k=1}^{m} X_k$ as the sequence of first m coordinates of each sequence in $\widetilde{t_m}$.

Finally, we define the sequence $\overline{p} = (p_n)$ as follows: given $n \in \mathbb{N}$, write $\widetilde{t_n} = (\overline{s_k})$ and define

$$p_n = \lim_{k \to \infty} \overline{s_k}_n$$

which exists by our observations above.

We claim that $\overline{p}$ is a cluster point for $(\overline{x_k})$.

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that

$$\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2}$$

Write $\widetilde{t_N} = (\overline{s_k})$. Then, $(\overline{s_k})$ converge pointwise in the first N coordinates to $(p_1, \ldots, p_N)$. By **Lemma** 2.2, ρ respects this pointwise convergence, so we can choose $K \in \mathbb{N}$ so that if $k > K$, then

$$\sum_{j=1}^{N} \frac{1}{2^j} \hat{d}_j\left(\overline{s_k}_{,j}, p_j\right) < \frac{\varepsilon}{2}$$

So, if $k > K$, we have

$$\begin{aligned}
\rho(\overline{s_k}, \overline{p}) &= \sum_{j=1}^{\infty} \frac{1}{2^j} \hat{d}\left(\overline{s_k}_{,j}, p_j\right) \\
&= \sum_{j=1}^{N} \frac{1}{2^j} \hat{d}\left(\overline{s_k}_{,j}, p_j\right) + \sum_{j=N+1}^{\infty} \frac{1}{2^j} \hat{d}\left(\overline{s_k}_{,j}, p_j\right) \\
&\leq \sum_{j=1}^{N} \frac{1}{2^j} \hat{d}\left(\overline{s_k}_{,j}, p_j\right) + \sum_{j=N+1}^{\infty} \frac{1}{2^j} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus, for any $\varepsilon > 0$, there are infinitely many elements in $(\overline{x_k})$ within a distance of ε from $\overline{p}$, so $\overline{p}$ is a cluster point.

So, any sequence in X has a cluster point, and hence X is compact. □

Lemma 2.4. *Let $(X, \|\cdot\|)$ be a separable normed vector space. Then the unit ball is separable with respect to the subspace topology.*

Proof. We let $\overline{B}(0,1)$ denote the closed unit ball of X. Let A be a countable dense subset of X, and consider the set $A' = A \cap \overline{B}(0,1)$. Countability of $A' \subseteq A$ is a clear consequence of countability of A.

We aim to show A' is dense in $\overline{B}(0,1)$. Let U be a non-empty open set in $\overline{B}(0,1)$. Then, by definition of the subspace topology, $U = \overline{B}(0,1) \cap U'$ for some open $U' \subseteq X$. Since $U \neq \emptyset$, we have some $x \in U$; in particular, we have $x \in \overline{B}(0,1)$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U'$.

Assume without loss of generality that $\varepsilon < 1$. Set $r = 1 - \frac{\varepsilon}{2}$ and choose by density of A some $a \in A \cap B(rx, 1 - r)$. Then, we have

$$\begin{aligned} \|a\| &\leq \|a - rx\| + \|rx\| \\ &< (1 - r) + r \\ &= 1 \end{aligned}$$

so, in fact, $a \in A'$. On the other hand,

$$\begin{aligned} \|a - x\| &\leq \|a - rx\| + \|rx - x\| \\ &< (1 - r) + |r - 1| \\ &= 2(1 - r) \\ &= \varepsilon \end{aligned}$$

so that $a \in B(x, \varepsilon) \subseteq U'$. In particular, $a \in \overline{B}(0, 1)$ and $a \in U'$, so $a \in U$.

Thus, for every non-empty open subset $U \subseteq \overline{B}(0, 1)$, the intersection $\overline{B}(0, 1) \cap A'$ is non-empty, so A' is dense in $\overline{B}(0, 1)$ with respect to the subspace topology.

□

§3 Proof of main result

Let $(X, \|\cdot\|)$ be a separable normed vector space. Denote by B the unit ball of X and by B^* the unit ball of X^* under the operator norm.

Let $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ be a dense subset of B, which exists by **Lemma** 2.4. We define a map $\mathbf{S} : B^* \to [-1, 1]^{\mathbb{N}}$ by

$$\mathbf{S}(\phi) = (\phi(\alpha_1), \phi(\alpha_2), \dots)$$

Since $\|\phi\|_{\text{op}} \leq 1$ and $\alpha_k \in B$, we have $|\phi(\alpha_k)| \leq 1$ for all $k \in \mathbb{N}$, so $\mathbf{S}$ is indeed well-defined.

Also, note that $\mathbf{S}$ is injective, for if $\mathbf{S}(\phi) = \mathbf{S}(\psi)$, then ϕ, ψ are uniformly continuous functions which agree on a dense subset of B and hence must agree on all of B. Since ϕ, ψ are linear, their equality on a neighborhood of 0 implies equality everywhere.

Recall that by **Lemma** 2.2, pointwise convergence in $[-1, 1]^{\mathbb{N}}$, where $[-1, 1]$ is considered with the standard euclidean metric, is metrizable with metric ρ. We take our metric $d : B^* \times B^* \to \mathbb{R}$ to be

$$d(\phi, \psi) = \rho(\mathbf{S}(\phi), \mathbf{S}(\psi))$$

Note that d inherits non-negativity, symmetry, and the triangle inequality from ρ. Moreover, by injectivity of $\mathbf{S}$, we have

$$d(\phi, \psi) = 0 \iff \mathbf{S}(\phi) = \mathbf{S}(\psi) \iff \phi = \psi$$

so d is positive-definite. Thus, d is indeed a metric.

We claim that d metrizes pointwise convergence; that is, $\phi_n \longrightarrow_d \phi$ if and only if $\phi_n(x) \to \phi(x)$ for all $x \in X$.

Proof. Let (ϕ_n) be a sequence in B^*.

$\Longrightarrow$ Suppose ϕ_n converge to ϕ under d.

Since ρ metrizes pointwise convergence of sequences, this means $\mathbf{S}(\phi_n)$ converges pointwise to $\mathbf{S}(\phi)$. In other words, $\phi_n(\alpha_k) \to \phi(\alpha_k)$ for all k, so ϕ_n converge pointwise to ϕ on α.

Now, we show ϕ_n converge pointwise to ϕ on all of X.

Let $x \in X$. If $x = 0$, then certainly $\phi_n(0) \to \phi(0)$, by linearity. Otherwise, set $x' = x/\|x\|$ so that $x' \in B$ and write $x' = \lim_{j\to\infty} \alpha_{k_j}$. Let $\varepsilon > 0$ be given. Since $\phi(\alpha_{k_j}) \to \phi(x')$ and $\alpha_{k_j} \to x'$, choose $j \in \mathbb{N}$ such that both $\left\|\alpha_{k_j} - x'\right\| < \varepsilon/3$ and $\left|\phi(\alpha_{k_j}) - \phi(x')\right| < \varepsilon/3$. By pointwise convergence on α, choose $N \in \mathbb{N}$ so that if $n > N$, then $\left|\phi_n(\alpha_{k_j}) - \phi(\alpha_{k_j})\right| < \varepsilon/3$.

Let $n > N$. Then,

$$\begin{aligned}
|\phi_n(x') - \phi(x')| &\leq \left|\phi_n(x') - \phi_n(\alpha_{k_j})\right| + \left|\phi_n(\alpha_{k_j}) - \phi(\alpha_{k_j})\right| + \left|\phi(\alpha_{k_j}) - \phi(x')\right| \\
&< \left|\phi_n(x') - \phi_n(\alpha_{k_j})\right| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \left|\phi_n(x' - \alpha_{k_j})\right| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\leq \|\phi_n\|_{\text{op}} \left\|x' - \alpha_{k_j}\right\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

So, $\phi_n(x') \to \phi(x')$ and hence, by linearity,

$$\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \|x\|\phi_n(x') = \|x\|\phi(x') = \phi(x)$$

Thus, convergence under ρ implies pointwise convergence.

$\Longleftarrow$ Suppose ϕ_n converge pointwise to ϕ.

Then, in particular, $\phi_n(\alpha_k) \to \phi(\alpha_k)$ for all $k \in \mathbb{N}$, so the sequences $\mathbf{S}(\phi_n)$ converge pointwise to $\mathbf{S}(\phi)$. But, ρ metrizes pointwise convergence, so

$$\lim_{n\to\infty} d(\phi_n, \phi) = \lim_{n\to\infty} \rho(\mathbf{S}(\phi_n), \mathbf{S}(\phi)) = 0$$

and hence ϕ_n converge to ϕ under d.

□

Next, we'll show that B^* is in fact compact under d, but first we'll need the following result.

Claim: (B^*, d) is complete.

Proof. Let (ϕ_n) be a Cauchy sequence in (B^*, d).

We claim that $\lim_{n\to\infty} \phi_n(x)$ exists for all $x \in X$.

Let $x \in X$. If $x = 0$, the claim is clearly true, so assume $x \neq 0$. Set $x' = x/\|x\|$. We will show that $(\phi_n(x'))$ converges. For this, it suffices to show the sequence is Cauchy, as $\mathbb{R}$ is complete.

So, let $\varepsilon > 0$ be given. Since $x' \in B$, we can write $x' = \lim_{j\to\infty} \alpha_{k_j}$. In particular, we can choose $j_0 \in \mathbb{N}$ so that $\left\|\alpha_{k_j} - \alpha_{k_{j_0}}\right\| < \varepsilon/3$ for all $j > j_0$.

Since (ϕ_n) is Cauchy under d, so is $\big(\phi_n(\alpha_{k_{j_0}})\big)$. Thus, we can choose $N \in \mathbb{N}$ so that if $n, m > N$, then $\big|\phi_n(\alpha_{k_{j_0}}) - \phi_m(\alpha_{k_{j_0}})\big| < \varepsilon/3$.

Let $n, m > N$. Then,

$$\begin{aligned}
|\phi_n(x') - \phi_m(x')| &= \left|\phi_n\left(\lim_{j\to\infty} \alpha_{k_j}\right) - \phi_m\left(\lim_{j\to\infty} \alpha_{k_j}\right)\right| \\
&= \lim_{j\to\infty} \big|\phi_n(\alpha_{k_j}) - \phi_m(\alpha_{k_j})\big| \\
&\le \lim_{j\to\infty} \big[\big|\phi_n(\alpha_{k_j}) - \phi_n(\alpha_{k_{j_0}})\big| + \big|\phi_n(\alpha_{k_{j_0}}) - \phi_m(\alpha_{k_{j_0}})\big| + \big|\phi_m(\alpha_{k_{j_0}}) - \phi_m(\alpha_{k_j})\big|\big] \\
&= \lim_{j\to\infty} \big[\big|\phi_n(\alpha_{k_j} - \alpha_{k_{j_0}})\big| + \big|\phi_n(\alpha_{k_{j_0}}) - \phi_m(\alpha_{k_{j_0}})\big| + \big|\phi_m(\alpha_{k_{j_0}} - \alpha_{k_j})\big|\big] \\
&\le \lim_{j\to\infty} \big[\|\phi_n\|_{\text{op}}\big\|\alpha_{k_j} - \alpha_{k_{j_0}}\big\| + \big|\phi_n(\alpha_{k_{j_0}}) - \phi_m(\alpha_{k_{j_0}})\big| + \|\phi_m\|_{\text{op}}\big\|\alpha_{k_j} - \alpha_{k_{j_0}}\big\|\big] \\
&\le \lim_{j\to\infty} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\le \varepsilon
\end{aligned}$$

Thus, $(\phi_n(x'))$ converges, and hence

$$\lim_{n\to\infty} \phi_n(x) = \|x\| \lim_{n\to\infty} \phi_n(x')$$

exists.

So, we define $\phi : X \to \mathbb{R}$ by

$$\phi(x) = \lim_{n\to\infty} \phi_n(x)$$

It is clear that ϕ is linear and that $\phi \in B^*$. Moreover, by construction, ϕ_n converge pointwise to ϕ and so $\phi_n \longrightarrow_d \phi$.

Thus, (B^*, d) is complete. □

Finally, we can show compactness of (B^*, d).

Proof. Since $[-1, 1]$ is compact, we have by **Lemma** 2.3 that $\big([-1, 1]^{\mathbb{N}}, \rho\big)$ is compact. Then, by definition of d, we have for all $\phi, \psi \in B^*$ that

$$d(\phi, \psi) = \rho(\mathbf{S}(\phi), \mathbf{S}(\psi))$$

so $\mathbf{S} : B^* \to [-1, 1]^{\mathbb{N}}$ is an isometry.

Since (B^*, d) is complete and completeness is an isometry invariant, it follows that $\mathbf{S}(B^*)$ is complete. Moreover, complete subspaces are necessarily closed, so $\mathbf{S}(B^*)$ must be a closed subspace of $[-1, 1]^{\mathbb{N}}$. Finally, by Proposition 1.16.15 (Gauvreau), $\mathbf{S}(B^*)$ is compact as it is a closed subspace of a compact space. Since isometries are injective, $\mathbf{S}^{-1} : \mathbf{S}(B^*) \to B^*$ exists and is also an isometry (Gauvreau 1.17.5). Importantly, $\mathbf{S}^{-1}$ is continuous, and so – by the Extreme Value Theorem – $\mathbf{S}^{-1}(\mathbf{S}(B^*)) = B^*$ is compact.

□

§4 Extending the metric

We have shown that pointwise convergence on B^* is metrizable. By a similar argument, pointwise convergence on any bounded subset of X^* can be metrized. A natural question is whether this boundedness assumption is required. That is, is it possible to metrize pointwise convergence on *all of* X^*?

Interestingly, when X has countable dimension (for example, when X is the space of real polynomials), the answer is yes, with a somewhat different metric. We have the following theorem:

Theorem 4.1. *Let X be a normed vector space. If X has countable dimension, then pointwise convergence can be metrized on all of X^*.*

Proof. Suppose X has countable dimension. If X has countably infinite dimension, let $\beta = \{\beta_k : k \in \mathbb{N}\}$ be a basis for X. Otherwise X is finite dimensional, say $\dim(X) = n$, so we let $\beta = \{\beta_k : k \in \mathbb{N}\}$ where $\{\beta_1, \ldots, \beta_n\}$ forms a basis for X, and $\beta_i = 0$ for $k > n$.

Define $\mathbf{S} : X^* \to \mathbb{R}^{\mathbb{N}}$ by

$$\mathbf{S}(\phi) = (\phi(\beta_1), \phi(\beta_2), \ldots)$$

By **Lemma** 2.2, pointwise convergence in $\mathbb{R}^{\mathbb{N}}$ is metrizable with metric ρ, so we define $d : X^* \times X^* \to \mathbb{R}$ by

$$d(\phi, \psi) = \rho(\mathbf{S}(\phi), \mathbf{S}(\psi))$$

Note that $\mathbf{S}$ is injective, since if $\mathbf{S}(\phi) = \mathbf{S}(\psi)$ then ϕ, ψ agree on all basis vectors and hence on every element of X. Thus, d is a metric.

Let (ϕ_n) be a sequence in X^*. By a similar argument as in §3, if ϕ_n converge pointwise to ϕ, then $\phi_n \longrightarrow_d \phi$.

Now, suppose ϕ_n converge to ϕ under d. Again, by a similar argument as in §3, ϕ_n converge pointwise to ϕ on every β_k. This allows us to extend pointwise convergence to all of X as follows.

Let $x \in X$ and write $x = c_1\beta_{k_1} + \cdots + c_n\beta_{k_j}$.

Let $\varepsilon > 0$ be given. Set $C = 1 + \max_{1 \le i \le j} |c_i|$. By pointwise convergence on β, choose for each $1 \le i \le j$, some $M_i \in \mathbb{N}$ so that if $n > M_i$, then $|\phi_n(\beta_{k_i}) - \phi(\beta_{k_i})| < \varepsilon/(Cj)$. Let $n > N$. Then,

$$\begin{aligned}
|\phi_n(x) - \phi(x)| &= \left|\sum_{i=1}^{j} c_i\big(\phi_n\big(\beta_{k_j}\big) - \phi\big(\beta_{k_j}\big)\big)\right| \\
&\le \sum_{i=1}^{j} |c_i|\big|\phi_n\big(\beta_{k_j}\big) - \phi\big(\beta_{k_j}\big)\big| \\
&\le \sum_{i=1}^{j} C\big|\phi_n\big(\beta_{k_j}\big) - \phi\big(\beta_{k_j}\big)\big| \\
&< \sum_{i=1}^{j} C\frac{\varepsilon}{Cj} \\
&= \varepsilon
\end{aligned}$$

Thus, d metrizes pointwise convergence on X^*. □

We believe the converse of **Theorem 4.1** is also true, giving the following conjecture

Conjecture 4.2. *Let X be a normed vector space. Then, pointwise convergence of sequences in X^* can be metrized if and only if X has countable dimension.*

While a little progress has been made, the conjecture remains unsolved. Note that it is very similar to a well-established result from topology that the *weak*-$*$ topology is not metrizable. In this topology, we generalize sequences to objects known as *nets* and force any such net of functions to converge pointwise. While it is true that any arbitrary topology is completely characterized by which nets converge and to what, it is *not* true that a topology is characterized by its convergent sequences. In particular, it is conceivable that a metric which metrizes the pointwise convergence of *sequences* of functionals might exist while still failing to metrize the *weak*-$*$ topology. For more information on the *weak*-$*$ topology and related results, refer to [1] and [2].

§5 Future Directions

In this paper, we focused on pointwise convergence of linear functionals. However, the notion of pointwise convergence is one which makes sense for functions between *any* metric spaces, so the question of metrizing pointwise convergence in this more general setting is still meaningful and yet unanswered.

The proofs presented here do not immediately generalize as we make explicit use of the vector space structure of X and the completeness of $\mathbb{R}$. However, we believe that the important properties granted by this structure can be phrased in the language of metric spaces.

Recall that a function f between metric spaces X and Y is said to be *Lipschitz* if there exists some constant $M \geq 0$ such that $d(f(a), f(b)) \leq Md(a, b)$ for all $a, b \in X$.

We say a family $\mathcal{F}$ of functions between X and Y is *uniformly Lipschitz* if there exists a constant $M \geq 0$ such that $d(f(a), f(b)) \leq Md(a, b)$ for all $f \in \mathcal{F}$ and $a, b \in X$.

Note, then, that the unit ball of the dual is uniformly Lipschitz with $M = 1$. With this language, we have the following conjecture:

Conjecture 5.1. *Let X, Y be metric spaces with X separable. Let $\mathcal{F}$ be a uniformly Lipschitz family of functions between X and Y. Then, pointwise convergence in $\mathcal{F}$ is metrizable. Moreover, if Y is compact, then $\mathcal{F}$ is compact under said metric.*

References

[1] Narici, Lawrence; Beckenstein, Edward (2011). Topological Vector Spaces. Pure and applied mathematics (Second ed.). Boca Raton, FL: CRC Press. ISBN 978-1584888666. OCLC 144216834.

[2] Megginson, Robert E. (1998), An introduction to Banach space theory, Graduate Texts in Mathematics, vol. 183, New York: Springer-Verlag, pp. xx+596, ISBN 0-387-98431-